Asymptotic stability of breathers in some Hamiltonian networks of weakly coupled oscillators

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5.9.12

Abstract

We consider a Hamiltonian chain of weakly coupled anharmonic oscillators. It is well known that if the coupling is weak enough then the system admits families of periodic solutions exponentially localized in space (breathers). In this paper we prove asymptotic stability in energy space of such solutions. The proof is based on two steps: first we use canonical perturbation theory to put the system in a suitable normal form in a neighborhood of the breather, second we use dispersion in order to prove asymptotic stability. The main limitation of the result rests in the fact that the nonlinear part of the on site potential is required to have a zero of order 8 at the origin. From a technical point of view the theory differs from that developed for Hamiltonian PDEs due to the fact that the breather is not a relative equilibrium of the system.

1 Introduction

In this paper we consider the dynamical system with Hamiltonian

$$H := \sum_{k\in\mathbb{Z}} \left[\frac{p_k^2+q_k^2}{2} + V(q_k)\right] + \frac{\epsilon}{2}\sum_{k\in\mathbb{Z}}(q_{k+1}-q_k)^2 \ , \tag{1.1}$$

where V is an analytic function having a zero of order at least 8 at the origin. In 1994 MacKay and Aubry [MA94] proved that if ϵ is small enough, then there exist periodic solutions which are exponentially localized in space (breathers).

The problem of stability of the breathers has attracted a lot of interest since the discovery of such objects and indeed linear stability has been rapidly obtained through signature theory (see [MS98]). Concerning the nonlinear stability, the only known result for Hamiltonian networks ensures stability over times exponentially long with $1/\epsilon$ [Bam96]. However the presence of dispersion suggests that the breathers should be asymptotic stable (see e.g. [Bam98]). (For nice reviews on breathers see [Aub97, FW98].)

In the present paper we actually prove that breathers are asymptotically stable, at least if the nonlinear part of the on site potential fulfills $V(q) = O(|q|^8)$ as $q \to 0$. More precisely we prove that if the initial datum is close in the energy norm to a breather, then the distance of the solution from the breathers, as a function of time, is small as an element of $L^q_t(\mathbb{R}, \ell^r)$. As usual (q, r) are admissible pairs (see eq.(2.3) below for a precise definition).

We emphasize that such a result is one of the few examples of asymptotic stability in Hamiltonian systems for object which are neither equilibria nor relative equilibria. As far as we know the only other known example is that of the solitary wave of the FPU system (see [FP02, FP04, HW08, Miz09, Miz11]). For the theory of asymptotic stability of equilibria or relative equilibria see e.g. [SW90, BP92, Sig93, SW99, Cuc01, GNT04, BC11, Bam11].

The proof consists of essentially 2 steps, the first one consists in using canonical perturbation theory in order to put the system in a suitable normal form. The second one consists in proving and exploiting suitable Strichartz estimates (following [GNT04, Miz08]) to get asymptotic stability.

The first step goes as follows: consider first the system with $\epsilon = 0$ and introduce action angle coordinates (I, α) for the zero-th oscillator, thus one is reduced to a perturbation of a Hamiltonian of the form

$$\mathfrak{h}_0(I) + \sum_{k \neq 0} \frac{p_k^2 + q_k^2}{2} \ , \tag{1.2}$$

with $\mathfrak{h}_0$ a suitable function. If the perturbation does not contain terms linear in (p, q) then the manifold $p = q = 0$ is invariant. So the idea is to iteratively eliminate from the perturbation the terms linear in such variables. Furthermore it also useful to eliminate the terms of order zero in p, q,which depend on the angle α conjugated to I. This is expected to be possible under the so called first Melnikov condition, namely

$$\omega_0 \neq 1/n \ , \quad \omega_0 := \frac{\partial \mathfrak{h}_0}{\partial I} \ , \quad n \in \mathbb{Z} \ .$$

However we have not been able to find rigorous results on this problem before the paper [Gio12] in which Giorgilli proved the convergence of the normal form in the case of Lyapunov periodic orbits. The method by Giorgilli is based on his previous work [Gio01] (an improvement of Cherry's theorem [Che]). Actually it consists of a careful analysis (and estimate) of the formal iterative procedure used to put the system in normal form, analysis which allows to prove the convergence of such an iterative procedure.

Here we use a variant of Giorgilli's method. Theorem 3.1 of the present paper differs from Giorgilli's one in the fact that we are here in an infinite dimensional context and we also need here to keep control of some weighted norms of the perturbation. Furthermore, we have to study quite explicitly the first two steps of the iterative procedure in order to have a precise description of the linearization of the Hamiltonian at the breather.

The dispersive step is more standard and consists of a variant of the theory of [KPS09], which in turn is based on the previous results [SK05], [KKK06], [PS08] (see also [CT09]) and on ideas by [Miz08]. The only difference with respect to such works rests in the fact that in our case the dispersion is of order ϵ and we need to keep into account the dependence of all the constants on ϵ, thus we repeat, when needed some steps of the proofs of such papers.

It is worth mentioning that the requirement of having a nonlinearity starting with high degree is present also in all the quoted papers and up to now there are no results on the case of analytic nonlinearities with a potential vanishing at an order smaller then 8. It is probably possible to weaken such a requirement by increasing the dimension of the lattice. We also remark that the extension of the normal form Theorem 3.1 to higher dimensions is straightforward, while the adaptation of the dispersive part requires probably some nontrivial work. Finally we point out that the theory of this paper can also be adapted to deal with the model of [Bam98] in which the on site potential does not contain the quadratic term.

The paper is organized as follows: in Sect. 2 we state the main result; in Sect. 3 we state and prove the normal form result; in Sect. 4 we deal with the dispersive part of the proof and conclude the proof of the main theorem; in Appendix A we prove some technical lemmas needed in the part on normal form; in Appendix B we give some technical lemmas needed for the dispersive part.

Acknowledgments. First I would like to thank A. Giorgilli for some discussions on normal form theory and for pointing to my attention his works. I also thank J. Villanueva and H. Broer for some information on the normal form problem, D. Pelinovski and A. Komech for some bibliographic indications on the dispersive behavior of lattices.

2 Statement of the main result

We first introduce action angle variables

$$(I, \alpha) \in \mathbb{R}_+ \times \mathbb{T}$$

(here $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the torus) for the zero-th oscillator. We recall that these variables are characterized by the following properties: I, α are canonically conjugated, α is an angle (i.e. $\alpha = \alpha + 2\pi$), and the one particle Hamiltonian is a function of I only,

$$\frac{p_0^2 + q_0^2}{2} + V(q_0) = \mathfrak{h}_0(I) \ , \tag{2.1}$$

with a suitable function $\mathfrak{h}_0$. We recall that, if V is analytic and fulfills $V(q) = O(|q|^3)$ then also the variables (I, α) are analytic in a domain of the form $(0, C) \times \mathbb{T}$, $C > 0$, and then also $\mathfrak{h}_0$ is analytic.

From now on we parametrize the phase space by using the coordinates (I, α, p, q), $p = (p_k)_{k\neq 0}$, $q = (q_k)_{k\neq 0}$. Furthermore we will use the collective notations $\xi \equiv (p, q)$ and $\zeta \equiv (I, \alpha, \xi)$.

We denote by ℓ^r_s the space of the sequences $q \equiv (q_k)$ such that

$$\|q\|_{\ell^r_s} := \left(\sum_k |q_k|^r \langle k \rangle^{rs} \right)^{1/r} , \quad s \in \mathbb{R} , \quad 1 \le r < \infty$$

is finite. As usual $\langle k \rangle := \sqrt{1+k^2}$, ℓ^∞_s is defined by the sup norm.

We will also denote by $\mathbf{l}^r_s := \ell^r_s \oplus \ell^r_s$. If $s = 0$ we will write $\ell^r_0 =: \ell^r$ and similarly for $\mathbf{l}^r$.

We will use also spaces with exponential weights: we fix once for all a positive β and consider the spaces ℓ^+, respectively ℓ^- of the sequences such that the norm

$$\|q\|^2_+ := \sum_k \mathrm{e}^{\beta|k|} |q_k|^2 \ , \ \text{respectively} \quad \|q\|^2_- := \sum_k \mathrm{e}^{-\beta|k|} |q_k|^2 \ , \tag{2.2}$$

is finite. We will also denote $\mathbf{l}^\pm := \ell^\pm \times \ell^\pm$.

Remark 2.1. We did not specify the range of the index k. Most of times it will run over $\mathbb{Z} - \{0\}$, but sometimes over the whole $\mathbb{Z}$. Every time this will be clear from the context. *Furthermore, by abuse of notion we will say that a phase point* $\zeta \equiv (I, \alpha, \xi) \in \mathbf{l}^r_s$ *(or* $\zeta \in \mathbf{l}^\pm$*) if* $\xi \in \mathbf{l}^r_s$ *(or* $\xi \in \mathbf{l}^\pm$*).*

Given two phase point $\zeta \equiv (I, \alpha, \xi)$ and $\zeta' \equiv (I', \alpha', \xi')$ we define their distance according to the different norms by

$$\mathrm{d}_{\mathbf{l}^r}(\zeta; \zeta') := \max \{|I - I'| \, ; |\alpha - \alpha'| \, ; \|\xi - \xi'\|_{\mathbf{l}^r}\}$$
$$\mathrm{d}_\pm(\zeta; \zeta') := \max \left\{|I - I'| \, ; |\alpha - \alpha'| \, ; \|\xi - \xi'\|_\pm\right\} \ .$$

Following [KT98] we say that a pair (q, r) is admissible if $q \ge 6$, $r \ge 2$ and

$$\frac{1}{q} + \frac{1}{3r} \le \frac{1}{6} \ . \tag{2.3}$$

All along the paper we will use the notation $a \preceq b$ to mean "there exists a constant C, independent of all the relevant quantities, such that $a \le Cb$". Sometimes, when needed or when interesting, we will write explicitly the constant.

Denote by $b_0(I, t)$ the family of periodic solutions of the system with $\epsilon = 0$ defined by

$$b_0(I, t) := (I, \omega_0 t + \alpha_0, 0) = (I(t), \alpha(t), \xi(t)) \ , \quad \omega_0 := \frac{\partial \mathfrak{h}_0}{\partial I}(I) \tag{2.4}$$

and by $\gamma_0 := \bigcup_t b_0(I, t)$ the corresponding trajectory, then the main result of the paper is the following Theorem.

Theorem 2.2. *Assume that V is analytic in a neighborhood of zero and that $V(q) = O(|q|^8)$ as $q \to 0$, assume also that there exist positive C_{ω_0}, and $\Delta_1 < \Delta_2$, such that the variables (I, α) are real analytic in $[\Delta_1, \Delta_2] \times \mathbb{T}$ and the following inequality holds*

$$\left|\omega_0(I) - \frac{1}{n}\right| \geq C_{\omega_0} \ , \quad \forall n \in \mathbb{Z} \ , \quad \forall I \in [\Delta_1, \Delta_2] \ , \tag{2.5}$$

then there exists $\epsilon_* > 0$, such that, for any $0 < \epsilon < \epsilon_*$ there exists a family of periodic solutions $b_\epsilon(\mathcal{I}, t)$, $\mathcal{I} \in [\Delta_1, \Delta_2]$, of the system (1.1), with trajectories $\gamma_\epsilon(\mathcal{I}) := \cup_t b_\epsilon(\mathcal{I}, t)$, having the following properties:

- *i) the distance between the unperturbed breather and the true breather is small: $\mathrm{d}_+(\gamma_\epsilon(\mathcal{I}); \gamma_0(\mathcal{I})) \preceq \sqrt{\epsilon}$,*
- *ii) the family $\gamma_\epsilon(\mathcal{I})$ is asymptotically stable. Precisely, fix $\delta > 1/2$, then the following holds true: there exists $\epsilon_\delta > 0$ such that, if $\epsilon < \epsilon_\delta$ and the initial datum ζ_0 fulfills*

$$\inf_{\mathcal{I} \in [\Delta_1, \Delta_2]} \mathrm{d}_{\mathbf{l}^2}(\zeta_0, \gamma_\epsilon(\mathcal{I})) =: \mu < \epsilon^\delta \ , \tag{2.6}$$

then there exists an analytic function $\mathcal{I}(t)$ s.t.

- *ii.1) for any admissible pair (q, r) the function $t \mapsto \mathrm{d}_{\mathbf{l}^r}(\zeta(t); \gamma_\epsilon(\mathcal{I}(t)))$ is of class L^q_t and fulfills*

$$\|\mathrm{d}_{\mathbf{l}^r}(\zeta(.); \gamma_\epsilon(\mathcal{I}(.)))\|_{L^q_t} \preceq \epsilon^{-1/q}\mu \ . \tag{2.7}$$

- *ii.2) $|\mathcal{I}(t) - \mathcal{I}(0)| \preceq \dfrac{\mu^2}{\epsilon^{1/2}}$ and $\mathcal{I}_\pm := \lim_{t \to \pm\infty} \mathcal{I}(t)$ exists.*

3 Construction of the breather and normal form close to it

In this section $\mathbf{l}^\pm$ will always denote the space of the **complex** sequences $\zeta = (I, \alpha, \xi)$ s.t. $\|\xi\|_\pm < \infty$.

3.1 Statement

The main result of this section is the following theorem.

Theorem 3.1. *Assume that V is analytic in a neighborhood of zero and $V(q) = O(|q|^4)$ as $q \to 0$, assume also that there exist positive C_{ω_0}, $\Delta_1 < \Delta_2$, such that such that the variables (I, α) are real analytic in $[\Delta_1, \Delta_2] \times \mathbb{T}$ and the following inequality holds*

$$\left|\omega_0(I) - \frac{1}{n}\right| \geq C_{\omega_0} \ , \quad \forall n \in \mathbb{Z} \ , \quad \forall I \in [\Delta_1, \Delta_2] \ , \tag{3.1}$$

then there exists $\epsilon_0 > 0$*, and* $\forall|\epsilon| < \epsilon_0$ *there exist complex neighborhoods* $\mathcal{U}^{\pm} \subset \mathbf{l}^{\pm}$ *of* $[\Delta_1, \Delta_2] \times \mathbb{T} \times \{0\}$ *and an analytic canonical transformation* $T : \mathcal{U}^{\pm} \to \mathbf{l}^{\pm}$ *leaving invariant the space of real sequences, with the following properties:*

i) there exists a positive K_1 *s.t.*

$$\mathcal{U}^+ \supset [\Delta_1 - \frac{1}{K_1}, \Delta_2 + \frac{1}{K_1}] \times \mathbb{T} \times \left\{ \xi \ : \ \|\xi\|_+ \le \frac{\sqrt{\epsilon}}{K_1} \right\} \ , \tag{3.2}$$

and

$$\mathcal{U}^- \supset [\Delta_1 - \frac{1}{K_1}, \Delta_2 + \frac{1}{K_1}] \times \mathbb{T} \times \left\{ \xi \ : \ \|\xi\|_- \le \frac{\sqrt{\epsilon}}{K_1} \right\} \ . \tag{3.3}$$

ii) the transformed Hamiltonian $H \circ T$ *has the form*

$$H \circ T = \mathfrak{h}(I) + H_L + \mathcal{V} + \mathcal{Z} \ , \tag{3.4}$$

where

ii.1)

$$H_L := \sum_{k \neq 0} \frac{p_k^2 + q_k^2}{2} + \epsilon \left[\sum_{k \neq -1, 0} \frac{(q_{k+1} - q_k)^2}{2} + q_{-1}^2 + q_1^2 \right] \ , \tag{3.5}$$

$$\mathcal{V}(q) := \sum_{k \neq 0} V(q_k) \ , \tag{3.6}$$

ii.2) $\mathfrak{h}(I)$ *is an analytic function of* I *fulfilling (with an* l*-dependent constant)*

$$\sup_{\mathcal{U}^-} \left| \frac{\partial^l (\mathfrak{h} - \mathfrak{h}_0)}{\partial I^l} \right| \preceq \sqrt{\epsilon} \ , \quad \forall l \ge 0 \ ;$$

ii.3) $\mathcal{Z}$ *is such that its Hamiltonian vector field* $X \equiv (X_I, X_\alpha, X_\xi)$ *is analytic as a map* $X : \mathcal{U}^- \to \mathbf{l}^+$ *and its components fulfill the following estimates*

$$\sup_{(I,\alpha,\xi) \in \mathcal{U}^-} |X_I(I, \alpha, \xi)| \preceq \epsilon^{1/2} \|\xi\|_-^2 \tag{3.7}$$

$$\sup_{(I,\alpha,\xi) \in \mathcal{U}^-} \|X_\xi(I, \alpha, \xi)\| \preceq \epsilon^{3/2} \|\xi\|_- \ . \tag{3.8}$$

iii) T *fulfills the estimates*

$$\sup_{(I,\alpha,\xi) \in \mathcal{U}^-} |T_I(I, \alpha, \xi) - I| \preceq \epsilon^{1/2} \tag{3.9}$$

$$\sup_{(I,\alpha,\xi) \in \mathcal{U}^-} |T_\alpha(I, \alpha, \xi) - \alpha| \preceq \epsilon^{1/2} \tag{3.10}$$

$$\sup_{(I,\alpha,\xi) \in \mathcal{U}^-} \|T_\xi(I, \alpha, \xi) - \xi\|_+ \preceq \epsilon$$

where T_I, T_α, T_ξ *are the different components of* T*.*

Remark 3.2. The Hamiltonian $H \circ T$ admits the invariant manifold $\xi = 0$ which is foliated in periodic orbits. In the original coordinates such periodic orbits are exponentially localized in space and in fact are the breathers by MacKay and Aubry. Theorem 3.1 also contains some information on the Hamiltonian close the breather, information which is crucial for proving asymptotic stability.

Remark 3.3. Since, for any $r \geq 1$, the embeddings

$$\mathbf{l}^+ \hookrightarrow \mathbf{l}^r \hookrightarrow \mathbf{l}^-$$

are continuous, the transformation T is analytic also as a map from $\mathbf{l}^r$ to itself.

3.2 Proof of theorem 3.1

Before starting the construction it is useful to make the following coordinate transformation:

$$\begin{aligned} z_k &= \frac{p_k + \mathrm{i} q_k}{\sqrt{2}} \\ w_k &= \frac{p_k - \mathrm{i} q_k}{\sqrt{2}} \end{aligned} \tag{3.11}$$

which transform the symplectic form to

$$\mathrm{d}I \wedge \mathrm{d}\alpha + \mathrm{i} \sum_k \mathrm{d}z_k \wedge \mathrm{d}w_k \ .$$

The transformation (3.11) only multiplies the norms by a constant, so it is enough to prove theorem 3.1 in the new variables. In this section (and in appendix A) we will denote by $\xi \equiv (z, w)$ the new complex variables; the collection of the variables (I, α) will be denoted by $x \equiv (I, \alpha)$.

In order to keep into account the different size of the different variables we proceed as follows: fix some positive constants R_α, R_I, and define $R_\xi := \sqrt{\epsilon}$, then given a point $\zeta \equiv (I, \alpha, \xi)$ we define its norms by

$$\langle | \zeta | \rangle_\pm := \max \left\{ \frac{|I|}{R_I}, \frac{|\alpha|}{R_\alpha}, \frac{\|\xi\|_\pm}{R_\xi} \right\} \ . \tag{3.12}$$

Sometimes we will also denote

$$\langle | \xi | \rangle_\pm := \frac{\|\xi\|_\pm}{R_\xi} \ . \tag{3.13}$$

The complex closed ball of radius R and center ζ in such topologies will be denoted by $B_\pm(R, \zeta)$.

Remark 3.4. This is an ϵ dependent norm. The dependence of all the constants on ϵ will be recorded, on the contrary the quantities R_α, R_I, will play no role and will be considered as fixed.

We will develop perturbation theory in a complex neighborhood of the domain

$$\mathcal{G} := [\Delta_1, \Delta_2] \times \mathbb{T} \times \{0\} \ni (I, \alpha, \xi) \ . \tag{3.14}$$

We fix once for all a positive R. For $\delta \in [0, 1)$ we denote

$$\mathcal{G}_\delta^{\pm} := \bigcup_{\zeta \in \mathcal{G}} B_{\pm}(\zeta, R(1-\delta)) \ . \tag{3.15}$$

We now define what we mean by normal form.

Definition 3.5. For some $1 > \delta \geq 0$, let $f = f(I, \alpha, \xi)$ be a Hamiltonian function analytic on $\mathcal{G}_\delta^{+}$. The function f will be said to be in normal form if $f(I, \alpha, 0) = 0$ and $\mathrm{d}_\xi f(I, \alpha, 0) \equiv 0$, where d_ξ is the differential with respect to ξ.

Remark 3.6. If a Hamiltonian function has the form

$$H = \mathfrak{h}(I) + \sum_{k \neq 0} z_k w_k + f$$

with f in normal form then the manifold $\xi = 0$ is invariant for the dynamics and is foliated in periodic orbits with frequency $\partial_I \mathfrak{h}(I)$.

To start with we introduce some notations.

Given a Hamiltonian function $f = f(I, \alpha, \xi)$ we will denote

$$f^{(0)}(I, \alpha) := f(I, \alpha, 0) \ , \quad \left\langle f^{(0)} \right\rangle (I) := \frac{1}{2\pi} \int_0^{2\pi} f^{(0)}(I, \alpha) \mathrm{d}\alpha \ , \tag{3.16}$$

$$f^{(1)}(I, \alpha, \xi) := \mathrm{d}_\xi f^{(1)}(I, \alpha, 0, 0)\xi \tag{3.17}$$

$$\equiv \sum_{k \neq 0} \left[\mathrm{d}_{z_k} f^{(1)}(I, \alpha, 0, 0) z_k + \mathrm{d}_{w_k} f^{(1)}(I, \alpha, 0, 0) w_k \right] \ , \tag{3.18}$$

$$f^{(2)} := f - f^{(0)} - f^{(1)} \ , \tag{3.19}$$

so that $f^{(2)}$ is in normal form.

Furthermore, for $f = f^{(1)}$ there exists a map $f^1(I, \alpha)$ such that

$$f^{(1)}(I, \alpha, \xi) = \left\langle f^1(I, \alpha); \xi \right\rangle := \sum_{k \neq 0} \left(f^1_{z,k} z_k + f^1_{w,k} w_k \right) \ , \tag{3.20}$$

where the scalar product is that of $\mathbf{l}^2$.

Given a Hamiltonian function χ on $\mathcal{G}_\delta^{\pm}$ we will denote by X_χ its Hamiltonian vector field, by $[X_\chi]_\alpha \equiv \frac{\partial \chi}{\partial I}$ its α component, and similarly all the other components.

It is useful to introduce the operator J (Poisson tensor) defined by

$$J \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -\mathrm{i}w \\ \mathrm{i}z \end{pmatrix} \ ,$$

so that, with the above notations

$$\left[X_{f^{(1)}}\right]_{\xi} \equiv Jf^1 \ .$$

To measure the size of the Hamiltonian vector fields of functions we will use the following norms

$$N_{\delta}^{\nabla}(\chi) := \frac{1}{R} max \left\{ \sup_{\zeta \in \mathcal{G}_{\delta}^{+}} \langle |X_{\chi}(\zeta)| \rangle_{+} , \sup_{\zeta \in \mathcal{G}_{\delta}^{-}} \langle |X_{\chi}(\zeta)| \rangle_{-} \right\} \ , \tag{3.21}$$

$$N_{\delta}^{\mathcal{S}}(\chi) := \frac{1}{R} \sup_{\zeta \in \mathcal{G}_{\delta}^{-}} \langle |X_{\chi}(\zeta)| \rangle_{+} \ . \tag{3.22}$$

Definition 3.7. A function whose Hamiltonian vector field is analytic as a map from $\mathcal{G}_{\delta}^{\pm}$ to $\mathbf{l}^{\pm}$ will be said to be of class $\mathcal{A}_{\delta}$. A function whose Hamiltonian vector field is analytic as a map from $\mathcal{G}_{\delta}^{-}$ to $\mathbf{l}^{+}$ will be said to be of class $\mathcal{S}_{\delta}$.

The key estimates which will be used in estimating the normal form are given in the following lemma.

Lemma 3.8. *Let $f \in \mathcal{S}_d$ and $g \in \mathcal{A}_d$ be analytic functions, then one has*

$$N_{d+d_1}^{\mathcal{S}}(\{f;g\}) \leq \frac{2}{d_1} N_{d}^{\mathcal{S}}(f)\, N_{d}^{\nabla}(g) \ . \tag{3.23}$$

$$N_{d}^{\mathcal{S}}\left(f^{(0)}\right) \leq N_{d}^{\mathcal{S}}(f) \ , \quad N_{d}^{\mathcal{S}}\left(f^{(1)}\right) \leq N_{d}^{\mathcal{S}}(f) \ , \tag{3.24}$$

$$N_{d+d_1}^{\mathcal{S}}\left(f^{(2)}\right) \leq \frac{1}{d_1^2} N_{d}^{\mathcal{S}}(f) \ , \tag{3.25}$$

$$N_{d}^{\mathcal{S}}\left(\left\{f^{(1)};g^{(2)}\right\}^{(1)}\right) \leq \frac{3}{1-d} N_{d}^{\mathcal{S}}\left(f^{(1)}\right) N_{d}^{\nabla}\left(g^{(2)}\right) \ . \tag{3.26}$$

The proof will be given in Appendix A.

In particular the estimate (3.26) in which there is no d at the denominator (but $1-d$, which is bounded away from zero) is the key for the convergence of the normal form procedure.

The canonical transformation increasing by one the order of the non normalized part of the Hamiltonian will be constructed as the Lie transform generated by auxiliary Hamiltonian functions of the form $\chi^{(1)} + \chi^{(0)} \in \mathcal{S}_d$ with some d.

Remark 3.9. Denote by Φ_{χ}^{t} the flow of the Hamiltonian vector field of a Hamiltonian function $\chi \in \mathcal{S}_d$, then by standard existence and uniqueness theory one has that, if $N_{d}^{\mathcal{S}}(\chi) \leq d_1$ with $0 \leq d_1 < 1-d$ then Φ_{χ}^{t} exists at least up to time 1, furthermore one has

$$\sup_{\zeta \in \mathcal{G}_{d+d_1}^{-}} \left\langle \left| \Phi_{\chi}^{t}(\zeta) - \zeta \right| \right\rangle_{+} \leq R N_{d}^{\mathcal{S}}(\chi) \ . \tag{3.27}$$

Remark 3.10. By standard Hamiltonian theory, for any smooth f one has

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \Phi_{\chi}^{t} = \{\chi; f\} \circ \Phi_{\chi}^{t} \ ,$$

thus, defining the sequence $f_{(l)}$ by

$$f_{(0)} := f \ , \quad f_{(l)} := \left\{\chi; f_{(l-1)}\right\} \ , \quad l \geq 1 \ ,$$

one has, for any $N \geq 0$

$$f \circ \phi_\chi^t = \sum_{l=0}^{N} \frac{f_{(l)}}{l!} + \int_0^1 \frac{(1-s)^N}{N!} f_{(N+1)} \circ \Phi_\chi^s \, \mathrm{d}s \ . \tag{3.28}$$

Lemma 3.11. *Let $\chi \in \mathcal{S}_d$, with $0 \leq d_1 < 1-d$ and let $f \in \mathcal{A}_d$; fix $0 < d_1 < (1-d)$, assume $N_d^{\mathcal{S}}(\chi) \leq d_1/3$, then one has*

$$N_{d+d_1}^{\mathcal{S}}\left(f \circ \Phi_\chi^1 - \sum_{j=0}^{N} \frac{1}{j!} f_{(j)}\right) \preceq \frac{1}{d_1^{N+1}} N_d^{\nabla}(f) \left(N_d^{\mathcal{S}}(\chi)\right)^{N+1} \ . \tag{3.29}$$

The proof will be given in Appendix A.

As usual, in order to find the generating function for the normalizing transformation one has to solve a cohomological equation, which in our case will have the form

$$\{H_{lin}; \chi\} = \Psi^{(0)} + \Psi^{(1)} \ , \tag{3.30}$$

where χ is the unknown, $\Psi^{(0)}$, $\Psi^{(1)}$ are given functions,

$$H_{lin}(I, \xi) := \mathfrak{h}(I) + \sum_{k \neq 0} z_k w_k \tag{3.31}$$

and $\mathfrak{h}$ is a function of the action I only. The last estimate we need before starting the recursive construction of the normal form is contained in the following lemma, which will proved in section A.

Lemma 3.12. *Let $1 > \delta \geq 0$ be given and consider the equation* (3.30)*. Assume that $\langle \Psi^{(0)} \rangle = 0$, that $\Psi^{(0)} \in \mathcal{S}_\delta$, $\Psi^{(1)} \in \mathcal{S}_\delta$ and $\mathfrak{h} \in \mathcal{S}_\delta$. Denote $\omega(I) := \frac{\partial \mathfrak{h}}{\partial I}$, and assume that on $\mathcal{G}_\delta^-$ one has*

$$\left|\omega(I) - \frac{1}{n}\right| \geq C_\omega > 0 \ , \quad \forall n \in \mathbb{Z} \ , \tag{3.32}$$

then the cohomological equation (3.30) *has a solution $\chi = \chi^{(0)} + \chi^{(1)}$ which fulfills*

$$N_\delta^{\mathcal{S}}\left(\chi^{(0)}\right) \leq \pi \left[\sup_{\mathcal{G}_\delta^-} \frac{1}{\omega(I)}\right] N_\delta^{\mathcal{S}}\left(\Psi^{(0)}\right) \preceq N_\delta^{\mathcal{S}}\left(\Psi^{(0)}\right) \ , \tag{3.33}$$

$$N_\delta^{\mathcal{S}}\left(\chi^{(1)}\right) \leq 2\pi \left[\sup_{\mathcal{G}_\delta^-} \frac{1}{\omega(I)\left|\sin\left(\frac{\pi}{\omega(I)}\right)\right|}\right] N_\delta^{\mathcal{S}}\left(\Psi^{(1)}\right) \preceq N_\delta^{\mathcal{S}}\left(\Psi^{(1)}\right) \ . \tag{3.34}$$

We now check the analyticity properties of the vector field of the Hamiltonian (1.1), which we rewrite as

$$\mathcal{H}_0 := H = H_0 + \mathcal{Z}_2 + \mathcal{R}_0^{(1)} + \mathcal{R}_0^{(0)} \tag{3.35}$$

$$H_0 := \mathfrak{h}_0(I) + \sum_{k\neq 0} z_k w_k \ , \tag{3.36}$$

$$\mathcal{Z}_2 := \epsilon\left[\sum_{k\neq -1,0} \frac{(q_{k+1}-q_k)^2}{2} + q_{-1}^2 + q_1^2\right] + \sum_{k\neq 0} V(q_k) \tag{3.37}$$

$$\mathcal{R}_0^{(1)} := -\epsilon q_0(I,\alpha)\left[q_{-1}+q_1\right] \ , \quad \mathcal{R}_0^{(0)} := \epsilon[q_0(I,\alpha)]^2 \ , \tag{3.38}$$

where $q_k = q_k(z_k, w_k)$ for $k \neq 0$.

Lemma 3.13. *There exists $\epsilon_{*1} > 0$ such that, if $|\epsilon| < \epsilon_{*1}$, then $\mathcal{Z}_2 \in \mathcal{A}_0$, while $\mathcal{R}_0^{(1)}, \mathcal{R}_0^{(0)}, \mathfrak{h}_0(I) \in \mathcal{S}_0$ and the following estimates hold*

$$N_0^{\nabla}\left(\mathcal{Z}_2\right) \preceq \epsilon \ , \quad N_0^{\mathcal{S}}\left(\mathcal{R}_0^{(0)}\right) \preceq \epsilon \ , \quad N_0^{\mathcal{S}}\left(\mathcal{R}_0^{(1)}\right) \preceq \sqrt{\epsilon} \ . \tag{3.39}$$

The very simple proof is left to the reader.

We proceed in constructing the canonical transformation putting the system in normal form. To this end we have to fix a sequence of domains in which the transformed Hamiltonians will be defined. Thus fix

$$\delta_j := \delta \mathrm{e}^{-j} \ , \quad \tilde{\delta}_r := \sum_{j=1}^{r} \delta_j \ , \quad \delta := (\mathrm{e}-1)/2 \ , \tag{3.40}$$

so that $\sum_{j\geq 1} \delta_j = 1/2$.

The first two steps of the normalizing procedure have to be performed in detail in order to keep the needed information on the linearization of the equations at the breather.

Lemma 3.14. *Assume that (3.32) holds, then there exist positive ϵ_{*2} such that, for any $|\epsilon| < \epsilon_{*2}$, there exists an analytic canonical transformation $T_2 : \mathcal{G}^-_{\tilde{\delta}_2} \to \mathcal{G}^-_0$ which restricts to an analytic transformation $T_2 : \mathcal{G}^+_{\tilde{\delta}_2} \to \mathcal{G}^+_0$ such that*

$$\mathcal{H}_2 := \mathcal{H}_0 \circ T_2 = H_2 + \mathcal{Z}_2 + \mathcal{R}_2 \tag{3.41}$$

$$H_2 = \mathfrak{h}_2(I) + \sum_{k\neq 0} z_k w_k \ , \quad \mathfrak{h}_2 = \mathfrak{h}_0 + h_2$$

and the following estimates hold

$$N^{\mathcal{S}}_{\tilde{\delta}_2}\left(\mathcal{R}_2\right) \preceq \epsilon^{3/2} \ , \quad N^{\mathcal{S}}_{\tilde{\delta}_2}\left(h_2\right) \preceq \epsilon \tag{3.42}$$

Furthermore one has $T_2 = (\,\mathbb{1} + \mathcal{T}_1)(\,\mathbb{1} + \mathcal{T}_2)$ with $\mathcal{T}_j : \mathcal{G}^-_{\tilde{\delta}_j} \to \mathcal{G}^+_{\tilde{\delta}_{j-1}}$ $(j = 1, 2)$ analytic and fulfilling

$$\sup_{\zeta\in\mathcal{G}^-_{\tilde{\delta}_j}} \left\langle|\mathcal{T}_j(\zeta)|\right\rangle_+ \preceq (\sqrt{\epsilon})^j \ . \tag{3.43}$$

Remark 3.15. The important fact is that, up to order $\epsilon^{3/2}$ there are no contributions correcting $\mathcal{Z}_2$, whose form is explicitly known.

Proof. We proceed in two steps, each one increasing by $\epsilon^{1/2}$ the order of the non normalized part of the Hamiltonian.

Let $\chi_1^{(1)}$ be the solution of the cohomological equation

$$\left\{H_0;\chi_1^{(1)}\right\}=\mathcal{R}_0^{(1)}\ ,$$

so that $N_0^{\mathcal{S}}\left(\chi_1^{(1)}\right)\preceq\sqrt{\epsilon}$. Use $\Phi^1_{\chi_1^{(1)}}$ to transform the Hamiltonian, then one has

$$\begin{aligned}
\mathcal{H}_{01}:=\mathcal{H}_0\circ\Phi^1_{\chi_1^{(1)}}&=H_0+\left\{\chi_1^{(1)};H_0\right\}+\frac{1}{2}\left\{\chi_1^{(1)};\left\{\chi_1^{(1)};H_0\right\}\right\}\\
&+\frac{1}{2}\int_0^1(1-s)^2\left\{\chi_1^{(1)};\left\{\chi_1^{(1)};\left\{\chi_1^{(1)};H_0\right\}\right\}\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&+\mathcal{Z}_2+\int_0^1\left\{\chi_1^{(1)};\mathcal{Z}_2\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&+\mathcal{R}_0^{(0)}+\int_0^1\left\{\chi_1^{(1)};\mathcal{R}_0^{(0)}\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&+\mathcal{R}_0^{(1)}+\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}+\int_0^1\left\{\chi_1^{(1)};\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&=H_0+\frac{1}{2}\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}\\
&+\int_0^1\left[1-\frac{(1-s)^2}{2}\right]\left\{\chi_1^{(1)};\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&+\mathcal{Z}_2+\int_0^1\left\{\chi_1^{(1)};\mathcal{Z}_2\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s\\
&+\mathcal{R}_0^{(0)}+\int_0^1\left\{\chi_1^{(1)};\mathcal{R}_0^{(0)}\right\}\circ\Phi^s_{\chi_1^{(1)}}\mathrm{d}s
\end{aligned}$$

By using (3.23) and lemma A.4 the seventh line, the integral at the eighth line and the integral at the ninth line have a norm which is estimated by a constant times $\epsilon^{3/2}$. It remains to estimate the Poisson bracket

$$\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}\equiv\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}^{(0)}+\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}^{(2)}\ .$$

By equations (3.23) and (3.24) the first term at r.h.s. has norm $N^{\mathcal{S}}_{\cdot}(.)$ of order ϵ and is independent of ξ. We are now going to prove that

$$N_{\delta_1}^{\mathcal{S}}\left(\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}^{(2)}\right)\preceq\epsilon^2\ . \tag{3.44}$$

Denote $f:=\left\{\chi_1^{(1)};\mathcal{R}_0^{(1)}\right\}^{(2)}$; using the further notation

$$\chi_1^{(1)}=\left\langle\chi^1;\xi\right\rangle\ ,\quad\mathcal{R}_0^{(1)}=\left\langle\mathcal{R}_0^1;\xi\right\rangle\ ,$$

one has

$$f = \left\langle \frac{\partial \chi^1}{\partial I}; \xi \right\rangle \left\langle \frac{\partial \mathcal{R}_0^1}{\partial \alpha}; \xi \right\rangle - \left\langle \frac{\partial \chi^1}{\partial \alpha}; \xi \right\rangle \left\langle \frac{\partial \mathcal{R}_0^1}{\partial I}; \xi \right\rangle \ , \tag{3.45}$$

so that

$$-\left[X_f\right]_I = \left\langle \frac{\partial^2 \chi^1}{\partial \alpha \partial I}; \xi \right\rangle \left\langle \frac{\partial \mathcal{R}_0^1}{\partial \alpha}; \xi \right\rangle + \text{similar terms} \ . \tag{3.46}$$

By the definition of $N_{.}^{\mathcal{S}}(.)$, one has

$$\frac{1}{RR_\xi} \sup_{\mathcal{G}_0^-} \left\| \chi^1(I, \alpha) \right\|_+ \leq N_0^{\mathcal{S}} \left(\chi_1^{(1)} \right) \ ,$$

and therefore on $\mathcal{G}_{\delta_1}^-$, by Cauchy estimate, one has

$$\frac{1}{RR_\xi} \left\| \frac{\partial^2 \chi^1}{\partial \alpha \partial I} \right\|_+ \leq \frac{2}{R^2 R_I R_\alpha \delta_1^2} N_0^{\mathcal{S}} \left(\chi_1^{(1)} \right) \ ,$$

which gives

$$\frac{1}{RR_\alpha} \left\| \frac{\partial^2 \chi^1}{\partial \alpha \partial I} \right\|_+ \leq \frac{2}{R^2 R_I R_\alpha^2 \delta_1^2} N_0^{\mathcal{S}} \left(\chi_1^{(1)} \right) R_\xi \ .$$

Inserting in (3.46) and taking into account that $R_\xi = \sqrt{\epsilon}$ one has

$$\sup_{\mathcal{G}_{\delta_1}^-} \frac{1}{RR_\alpha} \left| \left\langle \frac{\partial^2 \chi^1}{\partial \alpha \partial I}; \xi \right\rangle \right| \leq \frac{2}{R^2 R_I R_\alpha^2 \delta_1^2} N_0^{\mathcal{S}} \left(\chi_1^{(1)} \right) R_\xi^2 \preceq \epsilon \sqrt{\epsilon} \ . \tag{3.47}$$

In a similar way one gets

$$\left| \left\langle \frac{\partial \mathcal{R}_0^1}{\partial \alpha}; \xi \right\rangle \right| \preceq \epsilon^{3/2} \ .$$

from which one has that the first term of (3.46) is of order ϵ^3. All the other terms can be estimated similarly getting the wanted estimate for the α and the I components of the vector field.

Concerning the ξ component of the vector field one has

$$\left[X_f\right]_\xi = J \frac{\partial \chi^1}{\partial I} \left\langle \frac{\partial \mathcal{R}_0^1}{\partial I}; \xi \right\rangle + \text{similar terms} \ . \tag{3.48}$$

In particular, acting as above one immediately proves that on $\mathcal{G}_{\delta_1}^-$,

$$\left| \left\langle \frac{\partial \mathcal{R}_0^1}{\partial I}; \xi \right\rangle \right| \preceq \epsilon^{3/2} \ .$$

We now add the estimate of the derivative of χ^1:

$$\sup_{\mathcal{G}_{\delta_1}^-} \left\| J \frac{\partial \chi^1}{\partial I} \right\|_+ \leq \frac{1}{RR_I\delta_1} \sup_{\mathcal{G}_0^-} \left\| J\chi^1 \right\|_+ = \frac{1}{RR_I\delta_1} \sup_{\mathcal{G}_0^-} \left\| [X_{\chi^{(1)}}]_\xi \right\|_+ \preceq \epsilon^{1/2} R_\xi \ ,$$

dividing by RR_ξ, one gets that the norm $N^{\mathcal{S}}_{\delta_1}(.)$ of the first term of (3.48) is of order ϵ^2. Considering all the other terms one gets (3.44).

We have thus shown that after this transformation the Hamiltonian has the form

$$\mathcal{H}_1 = H_0 + \mathcal{Z}_2 + \mathcal{R}_1 + \tilde{\mathcal{R}}_2 \tag{3.49}$$

where

$$\mathcal{R}_1 \equiv \mathcal{R}_1^{(0)} := \mathcal{R}_0^{(0)} + \left\{ \chi_1^{(1)}; \mathcal{R}_0^{(1)} \right\}^{(0)} \quad , \quad N^{\mathcal{S}}_{\delta_1}\left(\tilde{\mathcal{R}}_2\right) \preceq \epsilon^{3/2} \tag{3.50}$$

We now perform the second step removing the part of $\mathcal{R}_1$ dependent on α. To this end define

$$\Psi_2 \equiv \Psi_2^{(0)} := \mathcal{R}_1 - \langle \mathcal{R}_1 \rangle \ ,$$

and define $\chi_2 \equiv \chi_2^{(0)}$ as the solution of the cohomological equation $\{H_0; \chi_2\} = \Psi_2$. Transforming $\mathcal{H}_1$ one gets

$$\begin{aligned} \mathcal{H}_2 := \mathcal{H}_1 \circ \Phi^1_{\chi_2} &= H_0 + \int_0^1 s \left\{ \chi_2; \Psi_2 \right\} \circ \Phi^s_{\chi_2} \mathrm{d}s \\ &+ \langle \mathcal{R}_1 \rangle + \mathcal{Z}_2 + \int_0^1 \left\{ \chi_2; \mathcal{Z}_2 \right\} \circ \Phi^s_{\chi_2} \mathrm{d}s \\ &+ \int_0^1 \left\{ \chi_2; \langle \mathcal{R}_1 \rangle \right\} \circ \Phi^s_{\chi_2} \mathrm{d}s + \tilde{\mathcal{R}}_2 \circ \Phi^1_{\chi_2} \ . \end{aligned}$$

Defining $h_2 := \langle \mathcal{R}_1 \rangle$, and $\mathcal{R}_2$ to be the sum of the various integrals and of $\tilde{\mathcal{R}}_2 \circ \Phi^1_{\chi_2}$ and estimating the different terms, one immediately gets the thesis. □

We are now ready to state the iterative lemma which is the heart of the proof.

Lemma 3.16. *(Iterative Lemma). Assume that on* $[\Delta_1, \Delta_2]$ *the non resonance condition* (3.1) *holds, then there exist positive constants* $\epsilon_*, C_1, C_2, C_3, C_4, K$, *such that the following holds true: for any* $r \geq 2$ *and any* ϵ *with* $|\epsilon| < \epsilon_*$ *there exists a canonical transformation* $T_r : \mathcal{G}^-_{\tilde{\delta}_r} \to \mathcal{G}^-_{\tilde{\delta}_{r-1}}$, *which restricts to an analytic transformation* $T_r : \mathcal{G}^+_{\tilde{\delta}_r} \to \mathcal{G}^+_{\tilde{\delta}_{r-1}}$ *s.t.*

$$\mathcal{H}_r := \mathcal{H}_0 \circ T_r = H_r + \mathcal{Z}_r + \mathcal{R}_r \ , \tag{3.51}$$

where

$$H_r := \mathfrak{h}_r(I) + \sum_{k \neq 0} z_k w_k \ , \quad \mathfrak{h}_r := \mathfrak{h}_0 + h_1 + ... + h_r \ , \tag{3.52}$$

$$\mathcal{Z}_r = \mathcal{Z}_2 + Z_3 + ... + Z_r \ , \tag{3.53}$$

$$T_r = (\,1\!\!1 + \mathcal{T}_1) \circ ... \circ (\,1\!\!1 + \mathcal{T}_r) \tag{3.54}$$

and Z_j is in normal form for all j's. The following estimates hold

$$N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathcal{R}_r) \le C_1 (K\sqrt{\epsilon})^{r+1} \tag{3.55}$$

$$N^{\mathcal{S}}_{\tilde{\delta}_j}(h_j) \le C_1 (K\sqrt{\epsilon})^{j} \tag{3.56}$$

$$N^{\mathcal{S}}_{\tilde{\delta}_j}(Z_j) \le \frac{C_2}{\delta_j^3} (K\sqrt{\epsilon})^{j} \tag{3.57}$$

$$\sup_{\zeta \in \mathcal{G}^-_{\tilde{\delta}_j}} \langle |\mathcal{T}_j(\zeta)| \rangle_+ \le C_3 (K\sqrt{\epsilon})^{j} \ . \tag{3.58}$$

Furthermore one has

$$N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathfrak{h}_r - \mathfrak{h}_0) \le C_4 \epsilon \tag{3.59}$$

$$N^{\nabla}_{\tilde{\delta}_r}(\mathcal{Z}_r) \le C_4 \epsilon < 1 \tag{3.60}$$

Proof. For $r = 2$ the lemma coincides with lemma 3.14. We assume it is true for some r and we prove it for $r + 1$.

Define

$$h_{r+1} := \langle \mathcal{R}^{(0)}_r \rangle \ , \tag{3.61}$$

$$\Psi_{r+1} := \mathcal{R}^{(0)}_r - \langle \mathcal{R}^{(0)}_r \rangle + \mathcal{R}^{(1)}_r \ . \tag{3.62}$$

So in particular h_{r+1} satisfies (3.56), and one has

$$N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\Psi^{(1)}_{r+1}\right) \le N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathcal{R}_r) \ , \quad N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\Psi^{(0)}_{r+1}\right) \le 2N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathcal{R}_r) \ , \tag{3.63}$$

Then we define χ_{r+1} to be the solution of

$$\{H_r, \chi_{r+1}\} = \Psi_{r+1} \ ; \tag{3.64}$$

remark that, by (3.59), provided ϵ is small enough (uniformly in r), $\omega_r := \frac{\partial \mathfrak{h}_r}{\partial I}$ satisfies (3.32) with a smaller constant C_ω independent of r. Therefore χ_r exists and fulfills

$$N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\chi^{(0)}_{r+1}\right) \preceq N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathcal{R}_r) \ , \quad N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\chi^{(1)}_{r+1}\right) \preceq N^{\mathcal{S}}_{\tilde{\delta}_r}(\mathcal{R}_r) \ , \tag{3.65}$$

with constants which are independent of r (as all the constants that will be suppressed using the symbol $\preceq$).

We define now $\mathcal{T}_{r+1} := \Phi^1_{\chi_{r+1}} - \mathbb{1}$ and

$$Z_{r+1} := \mathcal{R}^{(2)}_r + \left\{\chi^{(0)}_{r+1}; \mathcal{Z}_r\right\} + \left\{\chi^{(1)}_{r+1}; \mathcal{Z}_r\right\}^{(2)} \tag{3.66}$$

$$\mathcal{R}_{r+1} := \left\{\chi^{(1)}_{r+1}; \mathcal{Z}_r\right\}^{(1)} \tag{3.67}$$

$$+ \int_0^1 \left[\{\chi_{r+1}; \mathcal{R}_r\} + (1-s)\left[\{\chi_{r+1}; \{\chi_{r+1}; \mathcal{Z}_r\}\} - \{\chi_{r+1}; \Psi_{r+1}\}\right]\right] \circ \Phi^s_{\chi_{r+1}} \mathrm{d}s$$

so that $\mathcal{H}_{r+1}$ has the wanted form. We are now going to estimate the different terms in order to prove that the estimates (3.55)-(3.60) hold at level $r+1$.

Concerning Z_{r+1} we estimate the last term, which is the worst one:

$$N^{\mathcal{S}}_{\tilde{\delta}_{r+1}}\left(\left\{\chi^{(1)}_{r+1};\mathcal{Z}_r\right\}^{(2)}\right)\le\frac{4}{\delta^2_{r+1}}N^{\nabla}_{\tilde{\delta}_r+\delta_{r+1}/2}\left(\left\{\chi^{(1)}_{r+1};\mathcal{Z}_r\right\}\right) \tag{3.68}$$
$$\le\frac{4}{\delta^2_{r+1}}\frac{2}{\delta_{r+1}}N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\chi_{r+1}\right)N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)\preceq\frac{1}{\delta^3_{r+1}}N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)$$

Adding the other estimates one gets

$$N^{\mathcal{S}}_{\tilde{\delta}_{r+1}}\left(Z_{r+1}\right)\preceq\left(\frac{1}{\delta^2_{r+1}}+\frac{N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)}{\delta_{r+1}}+\frac{N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)}{\delta^3_{r+1}}\right)N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)\preceq\frac{1}{\delta^3_{r+1}}N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)\ , \tag{3.69}$$

which, provided one chooses C_2 to be C_1 times the constant not written in the last of (3.69) gives (3.57) at level $r+1$.

We come to $\mathcal{R}_{r+1}$. All the terms can be estimated in a straightforward way using lemmas 3.8, 3.11 and A.4 giving,

$$N^{\mathcal{S}}_{\tilde{\delta}_{r+1}}\left(\mathcal{R}_{r+1}\right)\preceq N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)\left[N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)+\frac{N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)}{\delta_{r+1}}+\frac{N^{\mathcal{S}}_{\tilde{\delta}_r}\left(\mathcal{R}_r\right)}{\delta^2_{r+1}}N^{\nabla}_{\tilde{\delta}_r}\left(\mathcal{Z}_r\right)\right]\ , \tag{3.70}$$

Calling $\mathcal{C}$ the constant making true (3.70) one has

$$N^{\mathcal{S}}_{\tilde{\delta}_{r+1}}\left(\mathcal{R}_{r+1}\right)\le\mathcal{C}C_1(K\sqrt{\epsilon})^{r+1}\left[C_4\epsilon+\frac{C_1(K\sqrt{\epsilon})^{r+1}}{\delta \mathrm{e}^{-(r+1)}}+\frac{C_1(K\sqrt{\epsilon})^{r+1}}{\delta^2\mathrm{e}^{-2(r+1)}}C_4\epsilon\right]\ . \tag{3.71}$$

Taking ϵ_* small enough one can make the square bracket smaller then $(C_4+C_1)\epsilon$, which shows that (3.55) is fulfilled at order $r+1$ if one defines $K:=\mathcal{C}(C_4+C_1)$. Remark that actually one increases the order of the perturbation by ϵ at every step, however we made the choice of estimating ϵ by $\sqrt{\epsilon}$ in order to be able to give a formulation of the theorem which is also valid in the case $r=1,2$.

All the other estimates are simpler and are omitted.

The key point in getting the estimate (3.55), which in turn is the key to get the convergence, rests in the fact that we separated from $\mathcal{R}_{r+1}$ the second two terms of (3.66) and furthermore in the fact that the first term of (3.67) fulfills the improved estimate (3.26). $\square$

Proof of Theorem 3.1. First remark that, due to the uniformity of the estimates, in Lemma 3.16 one can pass to the limit $r\to\infty$, getting a transformation $T:=T_\infty$ which is defined on $\mathcal{G}^{\pm}_{3/8}\subset\mathcal{G}^{\pm}_{1/2}$, which puts the system in normal form.

To get the estimate (3.10) remark first that, from (3.58), one has

$$\langle | \mathbb{1} - T| \rangle_+ \preceq \sum_{j\geq 1} (K\sqrt{\epsilon})^j \preceq \epsilon^{1/2} \ , \tag{3.72}$$

then (3.10) is just a component wise formulation of (3.72).

To estimate X, first remark that $\mathcal{Z} := \lim_{r\to\infty} \mathcal{Z}_r$ is defined and analytic in $\mathcal{G}^-_{3/8}$ and fulfills

$$N^{\mathcal{S}}_{\frac{3}{8}}(\mathcal{Z}) \preceq \sum_{j\geq 3} N^{\mathcal{S}}_{\frac{3}{8}}(\mathcal{Z}_j) \preceq \epsilon^{3/2} \ ,$$

which, written component wise, gives

$$\sup_{\zeta\in\mathcal{G}^-_{\frac{3}{8}}} |X_I(\zeta)| \preceq \epsilon^{3/2} \ , \tag{3.73}$$

$$\sup_{\zeta\in\mathcal{G}^-_{\frac{3}{8}}} |X_\alpha(\zeta)| \preceq \epsilon^{3/2} \ , \tag{3.74}$$

$$\sup_{\zeta\in\mathcal{G}^-_{\frac{3}{8}}} \|X_\xi(\zeta)\|_+ \preceq \epsilon^2 \ . \tag{3.75}$$

Since $\mathcal{Z}$ is in normal form one has $X_I(I,\alpha,0) = \mathrm{d}_\xi X_I(I,\alpha,0) = 0$, and thus, using the standard formula for the remainder of the Taylor expansion, one has

$$X_I(I,\alpha,\xi) = \int_0^1 (1-\tau)\mathrm{d}^2_\xi X_I(I,\alpha,\tau\xi)(\xi,\xi)\mathrm{d}\tau \ ;$$

Using the analyticity of X_I as a function of ξ in the domain $\|\xi\|_- \leq 3\epsilon^{1/2}/8$ one gets

$$\sup_{\zeta\in\mathcal{G}^-_{\frac{3}{8}}\cap\{\|\xi\|_-\leq\frac{\sqrt{\epsilon}}{4}\}} \left\|\mathrm{d}^2_\xi X_I(\zeta)\right\| \leq \frac{2}{(\sqrt{\epsilon}/4)^2} \sup_{\zeta\in\mathcal{G}_{\frac{3}{8}}} |X_I(\zeta)| \preceq \sqrt{\epsilon} \ ,$$

where the norm at the first term is for $\mathrm{d}^2_\xi X_I$ considered as quadratic form on the space of the ξ's endowed by the norm $\|.\|_-$. This proves (3.7).

Similarly, using

$$X_\xi(I,\alpha,\xi) = \int_0^1 \mathrm{d}_\xi X_\xi(I,\alpha,\tau\xi)\xi\mathrm{d}\tau$$

equation (3.75) and Cauchy estimate for the differential, one gets (3.8). □

4 Dispersive estimates

We first establish decay and Strichartz estimates for the group generated by the linear operator representing the first order normal form, namely for the flow of the linear system with Hamiltonian

$$H_L := \sum_{k\neq 0} \frac{p_k^2+q_k^2}{2} + \frac{\epsilon}{2}\sum_{k\neq -1,0} (q_{k+1}-q_k)^2 + \epsilon(q_1^2+q_{-1}^2) \ . \tag{4.1}$$

4.1 Linear local decay estimates

In order to prove the decay estimates it is useful to remark that the system (4.1) consists of two decoupled systems, the first consisting of the left hand part of the chain and having Hamiltonian

$$H_{L1} := \sum_{k \le -1} \frac{p_k^2 + q_k^2}{2} + \frac{\epsilon}{2} \sum_{k=-\infty}^{-1} (q_{k+1} - q_k)^2 \ , \tag{4.2}$$

where the phase space variables are $(p_k, q_k)_{k \le -1}$, while $q_0 \equiv 0$. Analogously the second system consists of the right hand part of the chain. Furthermore the system (4.2) can be viewed as the restriction of the system with Hamiltonian

$$H_S := \sum_{k \in \mathbb{Z}} \frac{p_k^2 + q_k^2}{2} + \frac{\epsilon}{2} \sum_{k \in \mathbb{Z}} (q_{k+1} - q_k)^2 \tag{4.3}$$

to skew symmetric sequences, namely the space of the sequences $(p_k, q_k)_{k \in \mathbb{Z}}$ such that $p_k = -p_{-k}$, $q_k = -q_{-k}$. The same is true for the system describing the right hand part of the chain.

Thus we start by establishing the needed decay estimates for the restriction of (4.3) to skew-symmetric sequences (actually when needed we will explicitly assume skew-symmetry of the sequences).

The system (4.3) is a Klein Gordon chain with small dispersion, so we actually follow the procedure of [SK05] and [KKK06] just keeping into account that we need estimates uniform in ϵ and that we are just interested in skew-symmetric sequences.

Consider the Hamilton equations of (4.3), namely

$$\begin{aligned} \dot{p}_k &= -q_k + \epsilon(q_{k+1} + q_{k-1} - 2q_k) \\ \dot{q}_k &= p_k \end{aligned} \tag{4.4}$$

and denote by $S_\epsilon^0(t)$ its evolution operator, namely the operator that to $\xi \equiv (p, q)$ associates the value at time t of the solution with initial datum ξ.

First remark that, by conservation of energy one has that, for ϵ small enough the system (4.4) is globally well posed in $\mathbf{l}^2$ and the inequality

$$\left\| S_\epsilon^0(t)\xi \right\|_{\mathbf{l}^2} \preceq \|\xi\|_{\mathbf{l}^2} \tag{4.5}$$

holds.

All along the proofs we will make use of the discrete Fourier transform defined by

$$q_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{q}(\theta) \mathrm{e}^{\mathrm{i}k\theta} \mathrm{d}\theta \ ,$$

with

$$\hat{q}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} q_k \mathrm{e}^{-\mathrm{i}k\theta} \ .$$

As usual the key property is that

$$(\Delta q)^\wedge(\theta) = -(2 - 2\cos\theta)\hat{q}(\theta) = -\left[4\sin^2\frac{\theta}{2}\right]\hat{q}(\theta) \ , \tag{4.6}$$

where

$$(\Delta q)_k := q_{k+1} + q_{k-1} - 2q_k$$

is the discrete Laplacian.

Lemma 4.1. *There exists ϵ_0 s.t., if $0 < \epsilon < \epsilon_0$ then the operator $S^0_\epsilon(t)$ fulfills*

$$\left\| S^0_\epsilon(t)\xi \right\|_{\mathbf{l}^\infty} \preceq \frac{1}{\langle t\epsilon\rangle^{1/3}} \ . \tag{4.7}$$

Proof. Rewrite (4.4) as a second order equation in Fourier coordinates, then it takes the form

$$\frac{d^2\hat{q}}{dt^2}(\theta) = -\nu(\theta)^2\hat{q}(\theta) \ , \quad \nu(\theta) := \sqrt{1 + 4\epsilon\sin^2\frac{\theta}{2}} \ , \tag{4.8}$$

whose solution is

$$\hat{q}(\theta, t) = \hat{q}(\theta, 0)\cos(\nu(\theta)t) + \frac{\hat{p}(\theta, 0)}{\nu(\theta)}\sin(\nu(\theta)t) \ ,$$

from which, returning to the space variables one gets

$$q_k(t) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\left[\hat{q}(\theta, 0)\cos(\nu(\theta)t) + \frac{\hat{p}(\theta, 0)}{\nu(\theta)}\sin(\nu(\theta)t)\right]\mathrm{e}^{\mathrm{i}k\theta}\mathrm{d}\theta$$

which is the linear combination of integrals of the form

$$\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat{q}(\theta, 0)\mathrm{e}^{\pm\mathrm{i}\nu(\theta)t + \mathrm{i}k\theta}\mathrm{d}\theta = \sum_{j\in\mathbb{Z}}\frac{q_j(0)}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\pm\mathrm{i}\nu(\theta)t + \mathrm{i}(k-j)\theta}\mathrm{d}\theta \tag{4.9}$$

and of the corresponding terms with p instead of q.

We now estimate the integral at r.h.s. using the Van der Corput Lemma (Lemma B.1 of the appendix). Writing $\varphi(\theta, \rho) := \nu(\theta) + \rho\theta$, one has

$$\sup_{k,j}\left|\int_{-\pi}^{\pi}\mathrm{e}^{\pm\mathrm{i}\nu(\theta)t + \mathrm{i}(k-j)\theta}\mathrm{d}\theta\right| \leq \sup_{\rho\in\mathbb{R}}\left|\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}\varphi(\theta,\rho)t}\mathrm{d}\theta\right| \ ,$$

where, for definiteness, we choosed the sign $+$. We split the interval of integration $[-\pi, \pi] = I_1 \cup I_2$ with

$$I_1 := \left[0, \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right] \cup \left[\frac{7\pi}{8}, \pi\right]$$
$$I_2 := \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \cup \left[\frac{5\pi}{8}, \frac{7\pi}{8}\right] \ ,$$

so that one has

$$|\varphi''(\theta,\rho)| = \left|\epsilon\cos 2\theta + O(\epsilon^2)\right| \geq C\epsilon\ ,\quad \forall\rho\in\mathbb{R}\ ,\quad \forall\theta\in I_1$$
$$|\varphi'''(\theta,\rho)| = \left|-2\epsilon\sin 2\theta + O(\epsilon^2)\right| \geq C\epsilon\ ,\quad \forall\rho\in\mathbb{R}\ ,\quad \forall\theta\in I_2\ .$$

Thus by Lemma B.1 one has

$$\sup_{\rho\in\mathbb{R}}\left|\int_{I_1} \mathrm{e}^{\mathrm{i}\varphi(\theta,\rho)}\mathrm{d}\theta\right| \preceq \frac{1}{|\epsilon t|^{1/2}}\ ,$$
$$\sup_{\rho\in\mathbb{R}}\left|\int_{I_2} \mathrm{e}^{\mathrm{i}\varphi(\theta,\rho)}\mathrm{d}\theta\right| \preceq \frac{1}{|\epsilon t|^{1/3}}\ ,$$

from which, using also

$$|q_k(t)| \leq \|q(t)\|_{\ell^2} \preceq \|\xi(0)\|_{\mathbf{l}^2} \preceq \|\xi(0)\|_{\mathbf{l}^1}\ ,$$

to control $t\to 0$, one gets

$$|q_k(t)| \preceq \|\xi(0)\|_{\mathbf{l}^1}\min\left\{1, \frac{1}{|\epsilon t|^{1/2}} + \frac{1}{|\epsilon t|^{1/3}}\right\} \preceq \frac{\|\xi(0)\|_{\mathbf{l}^1}}{\langle\epsilon t\rangle^{1/3}}\ .$$

Similarly, using

$$\hat{p}(\theta,t) = \hat{p}(\theta,0)\cos(\nu(\theta)t) - \hat{p}(\theta,0)\nu(\theta)\sin(\nu(\theta)t)\ ,$$

one gets the estimate of $|p_k(t)|$ and the proof of the Lemma. □

Next we need to establish weighted decay estimates.

In the following we will denote by $B(\mathbf{l}^2_s, \mathbf{l}^2_{-s})$ the space of bounded linear operators from $\mathbf{l}^2_s$ to $\mathbf{l}^2_{-s}$.

Lemma 4.2. *Let $s > 5/2$ then one has*

$$\left\|S^0_\epsilon(t)\xi\right\|_{\mathbf{l}^2_{-s}} \preceq \frac{1}{\langle\epsilon t\rangle^{3/2}}\|\xi\|_{\mathbf{l}^2_s}\ , \tag{4.10}$$

for all skew-symmetric sequence $\xi\in\mathbf{l}^2_s$.

Proof. We follow closely the procedure of [KKK06] and use their results (summarized in the appendix for the reader's convenience).

First rewrite (4.4) as $\mathrm{i}\dot{\xi} = A\xi$, where

$$A := \mathrm{i}\begin{bmatrix} 0 & -B \\ \mathbb{1} & 0\end{bmatrix}\ ,\quad B := \mathbb{1} - \epsilon\Delta\ ,$$

then it is easy to see that the spectrum $\sigma(A)$ of A is given by $\sigma(A) = I_+\cup I_-$ with $I_\pm := \pm[1, \sqrt{1+4\epsilon}]$

An explicit computation shows that the resolvent $R_A(\nu) := (A-\nu)^{-1}$ can be expressed in terms of the resolvent R_B of B as follows

$$R_A(\nu) = \begin{bmatrix}\nu R_B(\nu^2) & -\mathrm{i}(\mathbb{1} + \nu^2 R_B(\nu^2)) \\ \mathrm{i}R_B(\nu^2) & \nu R_B(\nu^2)\end{bmatrix}\ ,\quad \nu\notin\sigma(A) \tag{4.11}$$

Furthermore,remark that

$$R_B(\nu) = (1-\epsilon\Delta-\nu)^{-1} = \frac{1}{\epsilon}\left(-\Delta - \frac{\nu-1}{\epsilon}\right)^{-1} = \frac{1}{\epsilon}R_{-\Delta}\left(\frac{\nu-1}{\epsilon}\right) \ , \quad (4.12)$$

so that, from Lemma 3.1 of [KKK06] (see equation (B.3) below), the following limit exists in $B(\mathbf{l}^2_s, \mathbf{l}^2_{-s})$, $s > 1/2$

$$R^{\pm}_A := \lim_{\mu\to 0^+} R_A(\nu \pm \mathrm{i}\mu) \ , \nu \in I_- \cup I_+ \ .$$

Let $\Gamma_\pm$ be closed curves enclosing $I_\pm$ respectively, then by Cauchy theorem one has

$$S^0_\epsilon(t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_+\cup\Gamma_-} \mathrm{e}^{-\mathrm{i}t\nu} R_A(\nu)\mathrm{d}\nu \ .$$

We analyze the integral over Γ_-:

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_-} \mathrm{e}^{-\mathrm{i}t\nu} R_A(\nu)\mathrm{d}\nu = \frac{1}{2\mathrm{i}\pi}\int_{-1}^{-\sqrt{1+4\epsilon}} \mathrm{e}^{-\mathrm{i}t\nu}\left[R^+_A(\nu) - R^-_A(\nu)\right]\mathrm{d}\nu \ ;$$

making the change of variable $\nu = 1+\epsilon\omega$ and exploiting (4.12), one gets that such quantity coincides with

$$\frac{\mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}\pi}\int_0^{-\frac{\sqrt{1+4\epsilon}-1}{\epsilon}} \mathrm{e}^{-\mathrm{i}\epsilon t\omega}\times \quad (4.13)$$
$$\begin{bmatrix} (1+\epsilon\omega)[R^+_{-\Delta}(\varsigma(\omega)) - R^-_{-\Delta}(\varsigma(\omega))] & -\mathrm{i}(1+\omega\epsilon)^2[R^+_{-\Delta}(\varsigma(\omega)) - R^-_{-\Delta}(\varsigma(\omega))] \\ \mathrm{i}[R^+_{-\Delta}(\varsigma(\omega)) - R^-_{-\Delta}(\varsigma(\omega))] & (1+\epsilon\omega)[R^+_{-\Delta}(\varsigma(\omega)) - R^-_{-\Delta}(\varsigma(\omega))] \end{bmatrix}\mathrm{d}\omega$$

where

$$\varsigma(\omega) := \frac{(1+\epsilon\omega)^2-1}{\epsilon} = 2\omega + \epsilon\omega^2 \ .$$

Using $\overline{R^+_{-\Delta}(\varsigma)} = R^-_{-\Delta}(\varsigma)$, from which

$$\frac{R^+_{-\Delta}(\varsigma) - R^-_{-\Delta}(\varsigma)}{2\mathrm{i}} = \mathrm{Im}(R^+_{-\Delta}(\varsigma)) \ ,$$

one has that (4.13) coincides with

$$\frac{\mathrm{e}^{-\mathrm{i}t}}{\pi}\int_0^{-\frac{\sqrt{1+4\epsilon}-1}{\epsilon}} \mathrm{e}^{-\mathrm{i}\epsilon t\omega}\begin{bmatrix} 1+\epsilon\omega & -\mathrm{i}(1+\epsilon\omega) \\ \mathrm{i} & 1+\epsilon\omega \end{bmatrix} \quad (4.14)$$
$$\times\begin{bmatrix} \mathrm{Im}\, R^+_{-\Delta}(\varsigma(\omega)) & 0 \\ 0 & \mathrm{Im}\, R^+_{-\Delta}(\varsigma(\omega)) \end{bmatrix}\mathrm{d}\omega$$

Exploiting Lemma B.4 one verifies that we are now in the assumptions of Lemma B.5, which thus implies that the integral (4.14) is bounded by a constant times $|\epsilon t|^{-3/2}$. Treating in the same way the integral over Γ_+ one gets the result. □

Corollary 4.3. *Let $S_\epsilon(t)$ be the flow of the system with Hamiltonian* (4.1)*, then, for any $s > 5/2$, one has*

$$\|S_\epsilon(t)\xi\|_{\mathbf{l}^2} \preceq \|\xi\|_{\mathbf{l}^2} \ ' \tag{4.15}$$

$$\|S_\epsilon(t)\xi\|_{\mathbf{l}^\infty} \preceq \frac{\|\xi\|_{\mathbf{l}^1}}{\langle \epsilon t\rangle^{1/3}} \tag{4.16}$$

$$\|S_\epsilon(t)\xi\|_{\mathbf{l}^2_{-s}} \preceq \frac{\|\xi\|_{\mathbf{l}^2_s}}{\langle \epsilon t\rangle^{3/2}} \ . \tag{4.17}$$

4.2 Strichartz estimates

We first define the space-time norms which are needed in connections with Strichartz inequalities.

The space $L^q_{\epsilon t}([0,T],\mathbf{l}^r_s)$ is the space of the functions $F:[0,T]\to\mathbf{l}^r$ of class L^q endowed by the norm

$$\|F\|_{L^q_{\epsilon t}\mathbf{l}^r_s} := \left[\int_0^T \|F(t)\|^q_{\mathbf{l}^r}\,\epsilon \mathrm{d}t\right]^{1/q} = \epsilon^{1/q}\,\|F\|_{L^q_t\mathbf{l}^r_s} \ , \tag{4.18}$$

where the last norm is defined in the usual way. In most cases we will omit the indication of the interval of time and denote such a space simply by $L^q_{\epsilon t}\mathbf{l}^r$.

We use the result of [KT98] to get the Strichartz estimates for our model.

Lemma 4.4. *Let (q,r) and $(\tilde q,\tilde r)$ be admissible pairs, then the flow $S_\epsilon(t)$ of* (4.1) *fulfills*

$$\|S_\epsilon(t)\xi\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \left\|\mathbf{l}^2\right\| \tag{4.19}$$

$$\left\|\int_0^t S_\epsilon(t-\tau)F(\tau)\mathrm{d}\tau\right\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \frac{1}{\epsilon}\,\|F\|_{L^{\tilde q'}_{\epsilon t}\mathbf{l}^{\tilde r'}} \tag{4.20}$$

where $\tilde q'$ is such that $\frac{1}{\tilde q'}+\frac{1}{\tilde q}=1$ and similarly $\tilde r'$.

Proof. Since $S_\epsilon(t)$ is not unitary with respect to the norm of $\mathbf{l}^2$ we first modify the norm suitably. For $\xi\equiv(p,q)\in\mathbf{l}^2$ we define

$$\|\xi\|^2_{\mathbf{l}^2_B} := \langle p;p\rangle_{\ell^2} + \langle q;Bq\rangle_{\ell^2} \ , \tag{4.21}$$

where, as above, $B = \mathbb{1}-\epsilon\Delta$ and, in the second scalar product, one has to define $q_0\equiv 0$. It is immediate to verify that in this metric $S^*(t)=S(-t)$. Furthermore the norm (4.21) is equivalent to the standard norm of ℓ^2. Moreover the norm $\|Bq\|_{\ell^r_s}$ is equivalent to the norm $\|q\|_{\ell^r_s}$.

Before really starting with the proof remark that one has

$$\left\|f\left(\frac{\cdot}{\epsilon}\right)\right\|_{L^q_t} = \|f\|_{L^q_{\epsilon t}} \ ,$$

and that $S\left(\frac{\cdot}{\epsilon}\right)$ is a group fulfilling decay estimates independent of ϵ. Thus Theorem 1.2 of [KT98] directly applies giving (4.19). To get (4.20) one has

$$\begin{aligned}\left\|\int_0^t S_\epsilon(t-\tau)F(\tau)\mathrm{d}\tau\right\|_{L^q_{\epsilon t}\mathbf{l}^r} &= \left\|\int_0^{t/\epsilon} S_\epsilon\left(\frac{t}{\epsilon}-\tau\right)F(\tau)\mathrm{d}\tau\right\|_{L^q_t\mathbf{l}^r} \\ &= \frac{1}{\epsilon}\left\|\int_0^{t/\epsilon} S_\epsilon\left(\frac{t-\tau'}{\epsilon}\right)F\left(\frac{\tau'}{\epsilon}\right)\mathrm{d}\tau'\right\|_{L^q_t\mathbf{l}^r} \\ &\preceq \frac{1}{\epsilon}\left\|F\left(\frac{\cdot}{\epsilon}\right)\right\|_{L^{\tilde{q}'}_t\mathbf{l}^{\tilde{r}'}} = \frac{1}{\epsilon}\|F\|_{L^{\tilde{q}'}_{\epsilon t}\mathbf{l}^{\tilde{r}'}} \ .\end{aligned}$$

where the inequality is obtained by eq. (7) of [KT98]. □

Lemma 4.5. *Fix $s > 5/2$ then, for any admissible pair (q,r) one has*

$$\|S_\epsilon(t)\xi\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \|\xi\|_{\mathbf{l}^2} \ , \tag{4.22}$$

$$\left\|\int_0^t S_\epsilon(t-\tau)F(\tau)\ d\tau\right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \frac{1}{\epsilon}\|F\|_{\mathbf{l}^1_s L^2_{\epsilon t}} \ , \tag{4.23}$$

$$\left\|\int_0^t S_\epsilon(t-\tau)F(\tau)\ d\tau\right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \frac{1}{\epsilon}\|F\|_{L^1_{\epsilon t}\mathbf{l}^2} \ , \tag{4.24}$$

$$\left\|\int_0^t S_\epsilon(t-\tau)F(\tau)\ d\tau\right\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \frac{1}{\epsilon}\|F\|_{L^2_{\epsilon t}\mathbf{l}^2_s} \tag{4.25}$$

Proof. The proof is a minimal variation of Lemma 6 of [KPS09]. We begin by (4.23). This is the equivalent of equation (27) of [KPS09]. Since (27) is a consequence of the local decay estimate, eq. (4.10) with $\epsilon = 1$ implies, by the procedure of [KPS09], the validity of (4.23) in the case $\epsilon = 1$. The case with $\epsilon \neq 0$ is an immediate consequence of the same scaling argument used in the proof of Lemma 4.4.

Equation (4.22) follows by the TT^* argument when one considers $T : \ell^2 \to \mathbf{l}^\infty_{-s}L^2_{\epsilon t}$. (Remark that (4.22) is weaker then the corresponding equation in [KPS09], namely (25).)

Equations (4.24) and (4.25) follow from the previous ones by repeating exactly the argument in the proof of Lemma 6 of [KPS09]. □

4.3 Nonlinear estimates

Here we prove the following Lemma:

Lemma 4.6. *Fix $\delta > 1/2$, then there exists $\epsilon_\delta > 0$ s.t., if $0 < \epsilon < \epsilon_\delta$ then the following holds true. Let $(I(t),\alpha(t),\xi(t))$ be a solution of the Hamiltonian system* (3.4) *with initial datum $(I_0,\alpha_0,\xi_0) \in [\Delta_1 - \frac{1}{2K_1}, \Delta_2 + \frac{1}{2K_1}] \times \mathbb{T} \times \mathbf{l}^2$ s.t.*

$$\mu := \|\xi_0\|_{\mathbf{l}^2} < \epsilon^\delta \ , \tag{4.26}$$

then, for any admissible pair (q,r) and any $s>5/2$ one has

$$\|\xi\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \mu \ , \tag{4.27}$$

$$\|\xi\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \mu \ . \tag{4.28}$$

Furthermore the limit $I_\pm := \lim_{t\to\pm\infty} I(t)$ exists and fulfills

$$|I_\pm - I_0| \preceq \frac{\mu^2}{\epsilon^{1/2}} \ . \tag{4.29}$$

Proof. We proceed by "induction" as in [GNT04]: we are going to prove that, if the solution fulfills

$$\|\xi\|_{L^q_{\epsilon t}([0,T],\mathbf{l}^r)} \le M_1\mu \ , \tag{4.30}$$

$$\|\xi\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}[0,T]} \le M_2\mu \ , \tag{4.31}$$

then it belongs to the interior of the domain of definition of the transformation of Thoeorem 3.1, see item i), and furthermore for a suitable choice of M_1, M_2 and for ϵ small enough, the inequalities (4.30) and (4.31) hold with M_1, M_2 replaced by $M_1/2$ and $M_2/2$.

Using Duhamel formula rewrite the equation for ξ in the form

$$\xi(t) = S_\epsilon(t)\xi_0 + \int_0^t S_\epsilon(t-\tau)X_\xi(I(\tau),\alpha(\tau),\xi(\tau))\mathrm{d}\tau \\ + \int_0^t S_\epsilon(t-\tau)\left[X_{\mathcal{V}}(\xi(\tau))\right]\xi\mathrm{d}\tau \ . \tag{4.32}$$

We begin by estimating the norm $L^q_{\epsilon t}\mathbf{l}^r$ (where we omitted the interval of time). The first term at r.h.s. is estimated using (4.19) by

$$\|S_\epsilon(t)\xi_0\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \|\xi_0\|_{\mathbf{l}^2} = \mu \ .$$

For the second term, using (4.25) one has

$$\left\|\int_0^t S_\epsilon(t-\tau)X_\xi(I(\tau),\alpha(\tau),\xi(\tau))\mathrm{d}\tau\right\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \frac{1}{\epsilon}\|X_\xi(I,\alpha,\xi)\|_{L^2_{\epsilon t}\mathbf{l}^2_s}$$
$$\preceq \frac{1}{\epsilon}\|X_\xi(I,\alpha,\xi)\|_{L^2_{\epsilon t}\mathbf{l}^\infty_{s'}} \preceq \frac{1}{\epsilon}\epsilon^{3/2}\|\xi\|_{L^2_{\epsilon t}\mathbf{l}^\infty_{-s''}} \preceq \epsilon^{1/2}\|\xi\|_{L^2_{\epsilon t}\mathbf{l}^2_{-s''}} \ ,$$

where $s' > s+1/2$ and we used (3.8) for the third inequality, which is valid for any $s''>0$. Using Lemma B.6 the last quantity is smaller then

$$\epsilon^{1/2}\|\xi\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \le \epsilon^{1/2}M_2\mu \ ,$$

provided $s'' > s+1/2$.

For the third term one has, using (4.20),

$$\left\| \int_0^t S_\epsilon(t-\tau)\left[X_{\mathcal{V}}(\xi(\tau))\right]\xi \mathrm{d}\tau \right\|_{L^q_{\epsilon t}\mathbf{l}^r} \preceq \frac{1}{\epsilon}\left\|X_{\mathcal{V}}(\xi)\right\|_{L^1_{\epsilon t}\mathbf{l}^2}$$

$$\preceq \frac{1}{\epsilon}\left\|\xi\right\|^7_{L^7_{\epsilon t}\mathbf{l}^{14}} \preceq \frac{\mu^7}{\epsilon} M_1^7 \ , \tag{4.33}$$

where we used, for $p = 7$, the following inequalities

$$\left\|X_{\mathcal{V}}(\xi)\right\|_{L^1_{\epsilon t}\mathbf{l}^2} = \int_0^T \left[\sum_{k\neq 0}(V'(q_k(t)))^2\right]^{1/2} \epsilon \mathrm{d}t$$

$$\preceq \int_0^T \left[\sum_{k\neq 0}|q_k(t)|^{2p}\right]^{1/2} \epsilon \mathrm{d}t = \int_0^T \left\|q(t)\right\|^p_{\mathbf{l}^{2p}} \epsilon \mathrm{d}t = \left\|\xi\right\|_{L^p_{\epsilon t}\mathbf{l}^{2p}} \ ,$$

and the fact that $(p, 2p) = (7, 14)$ is an admissible pair.

Thus we have that the considered solution fulfills the inequality (4.30) with M_1 replaced by $M_1/2$ if the following inequality holds

$$1 + M_2\epsilon^{1/2} + \frac{M_1^7\mu^6}{\epsilon} \leq \frac{M_1}{C} \tag{4.34}$$

with a given large C.

We estimate now $\|\xi\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}}$. Making again reference to equation (4.32), by equation (4.22) one has

$$\left\|S_\epsilon(t)\xi_0\right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \left\|\xi_0\right\|_{\mathbf{l}^2} \preceq C_0\mu \ .$$

Then, by (4.23) one has

$$\left\| \int_0^t S_\epsilon(t-\tau)X_\xi(I(\tau),\alpha(\tau),\xi(\tau))\mathrm{d}\tau \right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}}$$

$$\preceq \frac{1}{\epsilon}\left\|X_\xi(I,\alpha,\xi)\right\|_{\mathbf{l}^1_s L^2_{\epsilon t}} \preceq \frac{1}{\epsilon}\left\|X_\xi(I,\alpha,\xi)\right\|_{L^2_{\epsilon t}\mathbf{l}^2_{s'}} \tag{4.35}$$

where $s' > s + 1/2$ (the proof of the last inequality is almost identical to the proof of Lemma B.6 and is omitted). Eq. (4.35) is estimated by using (3.8) and lemma B.6:

$$(4.35) \preceq \frac{\epsilon^{3/2}}{\epsilon}\left\|\xi\right\|_{L^2_{\epsilon t}\mathbf{l}^2_{-s''}} \preceq \epsilon^{1/2}\left\|\xi\right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \leq M_2\epsilon^{1/2}\mu \ .$$

The last term is estimated by eq.(4.24), which gives, like in (4.33),

$$\left\| \int_0^t S_\epsilon(t-\tau)\left[X_{\mathcal{V}}(\xi(\tau))\right]\xi \mathrm{d}\tau \right\|_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \frac{1}{\epsilon}\left\|X_{\mathcal{V}}(\xi)\right\|_{L^1_{\epsilon t}\mathbf{l}^2}$$

$$\preceq \frac{\mu^7}{\epsilon} M_1^7 \ ,$$

so that the considered solution fulfills the inequality (4.31) with M_2 replaced by $M_2/2$ if the following inequality holds

$$1 + M_2\epsilon^{1/2} + \frac{M_1^7\mu^6}{\epsilon} \leq \frac{M_2}{C} \tag{4.36}$$

with a given large C.

Now it is clear that both (4.34) and (4.36) are fulfilled if M_1 and M_2 are chosen strictly larger then C, with C the constant in (4.34) and (4.36), and ϵ is small enough. In particular this implies that also μ^6/ϵ is small.

Remark that from these inequalities it also follows that $\|\xi(t)\|_{\mathbf{l}^2} < \sqrt{\epsilon}/2K_1$, so that ξ is in the domain of validity of the normal form.

Concerning I, one has

$$I(t) = I_0 + \int_0^t X_I(\zeta(\tau))\mathrm{d}\tau \ .$$

One has

$$\int_0^t |X_I(\zeta(\tau))| \,\mathrm{d}\tau \preceq \frac{\epsilon^{1/2}}{\epsilon}\int_0^t \|\xi(\tau)\|^2_{\mathbf{l}^2_{-s}}\,\epsilon\mathrm{d}\tau = \frac{1}{\epsilon^{1/2}}\|\xi\|^2_{L^2_{\epsilon t}\mathbf{l}^2_{-s}}, \tag{4.37}$$

$$\preceq \frac{1}{\epsilon^{1/2}}\|\xi\|^2_{\mathbf{l}^\infty_{-s}L^2_{\epsilon t}} \preceq \frac{\mu^2}{\epsilon}M_2^2 \ , \tag{4.38}$$

which implies $|I(t) - I(0)| \preceq \epsilon^{2\delta-1}$ and therefore also $I(t)$ is close to I_0 and therefore, if ϵ is small enough $I(t) \in [\Delta_1 - \frac{3}{4K_1}, \Delta_2 + \frac{3}{4K_1})$ which is contained in the domain of validity of the normal form.

This allows to extend the estimates to $T = \infty$. From (4.38) follows that the integral (4.37) converges, and therefore the limit of $I(t)$ exists and (4.29) holds. $\square$

Proof of Theorem 2.2. First, the existence of the breather and item i) are a consequence of theorem 3.1 (see Remark 3.2).

Item ii.1 follows immediately by defining $\mathcal{I}$ as the action variable I in the coordinates introduced by the normal form theorem.

Finally, to get (2.7), remark that, in the coordinates introduced by Theorem 3.1

$$d_{\mathbf{l}^r}(\gamma_\epsilon(\mathcal{I}(t)); \zeta(t)) = \|\xi(t)\|_{\mathbf{l}^r} \ ,$$

then 2.7 follows from (4.27) and the fact that the canonical transformation T is Lipschitz in the $\mathbf{l}^r$ metric (see Remark 3.3), and therefore only multiplies distances by a number (which is of order 1 in our case). $\square$

A Technical lemmas for the normal form

We begin by the different estimates involved in Lemma 3.8.

The estimate (3.23) of the Poisson brackets coincides with that given in [BG93], lemma 5.2. For the sake of completeness we repeat here the argument of that paper.

Lemma A.1. *Let $g \in \mathcal{A}_d$ and $f \in \mathcal{S}_d$ be two functions with analytic vector field; then for any $d_1 < 1-d$, $\{g;f\} \in \mathcal{S}_{d+d_1}$ satisfies the inequality* (3.23).

Proof. First remark that

$$X_{\{f;g\}} = [X_f; X_g] = \mathrm{d}X_f\, X_g - \mathrm{d}X_g\, X_f\ . \tag{A.1}$$

Using Cauchy estimate one immediately has that, on $\mathcal{G}^-_{d+d_1}$, the norm of $\mathrm{d}X_f$ as a linear operator from $\mathbf{l}^-$ to $\mathbf{l}^+$ is smaller then $N^{\mathcal{S}}_d(f)/d_1$. It follows that the norm $N^{\mathcal{S}}_{d+d_1}(.)$ of first term of (A.1) is bounded by

$$\frac{1}{d_1} N^{\mathcal{S}}_d(f)\, N^{\nabla}_d(g)\ .$$

The second term is bounded in a similar way getting the thesis. □

Lemma A.2. *Let $f \in \mathcal{S}_d$ be a function with analytic vector field; let $0 < d_1 < 1-d$, then the estimates* (3.24) *and* (3.25) *hold.*

Proof. The estimate is trivial for the $^{(0)}$ component. Indeed the vector field of $f^{(0)}$ coincides with the value at $\xi = 0$ of the components (I,α) of the vector field of f.

We come to the estimate of the vector field of $f^{(1)}(I,\alpha,\xi) \equiv \mathrm{d}_\xi f(I,\alpha,0)\xi$. Remark that one has

$$\left[X_{f^{(1)}}\right]_\xi (I,\alpha) = \left[X_f\right]_\xi (I,\alpha,0)\ , \tag{A.2}$$

$$\left[X_{f^{(1)}}\right]_x (I,\alpha,\xi) = \mathrm{d}_\xi \left[X_f\right]_x (I,\alpha,0)\xi\ . \tag{A.3}$$

The estimate of (A.2) is straightforward. Concerning the estimate of (A.3), remark that, by Cauchy inequality one has

$$\left\| \mathrm{d}_\xi \left[X_f\right]_x (I,\alpha,0) \right\| \le \frac{1}{1-d} N^{\mathcal{S}}_d(f)\ , \tag{A.4}$$

and that, on $\mathcal{G}^-_d$ the norm (3.13) of ξ is smaller then $R(1-d)$. Thus one gets also the second of (3.24).

We come to the estimate of $f^{(2)}$. The components of its vector field are remainders of Taylor expansions truncated at suitable order of the components of X_f. In particular the term of higher order is in the x components. From standard formulae of the remainder of Taylor expansions one has

$$\left[X_{f^{(2)}}\right]_x (I,\alpha,\xi) = \int_0^1 (1-s)\mathrm{d}^2_\xi \left[X_f\right]_x (I,\alpha,s\xi)(\xi,\xi)\mathrm{d}s\ ;$$

using Cauchy estimate to estimate the norm of the second differential one gets that the argument of the integral, in $\mathcal{G}^-_{d+d_1}$ is estimated by

$$\frac{2}{R^2 d_1^2} R N^{\mathcal{S}}_d(f) \left[R(1-d-d_1)\right]^2\ ,$$

which, integrating and dividing by R in order to get the norm $N^{\mathcal{S}}_{d+d_1}(.)$ gives the result. $\square$

Lemma A.3. *Let $f = f^{(1)} \in \mathcal{S}_d$ and $g = g^{(2)} \in \mathcal{A}_d$. Then* (3.26) *holds.*

Proof. First remark that, denoting by $g_2(I,\alpha,\xi) := [\mathrm{d}^2_\xi g(I,\alpha,0)](\xi,\xi)$ the part of $g^{(2)}$ homogeneous of degree 2, one has

$$\left\{f^{(1)}; g^{(2)}\right\}^{(1)} = \left\{f^{(1)}; g_2\right\}^{(1)} \ .$$

So we first study g_2. Remark that, by a procedure similar to the one used in the proof of lemma A.2, one has

$$N^\nabla_d\,(g_2) \le N^\nabla_d\left(g^{(2)}\right) \ . \tag{A.5}$$

Denote $B(I,\alpha) := J^{-1}\mathrm{d}_\xi\left[X_{g^{(2)}}\right]_\xi(I,\alpha,0)$, where J is the Poisson tensor, then one has

$$g_2(I,\alpha,\xi) = \frac{1}{2}\left\langle \xi; B(I,\alpha)\xi\right\rangle \ , \quad \|B\| \le \frac{1}{1-d}N^\nabla_d\left(g^{(2)}\right) \ , \tag{A.6}$$

and furthermore B is symmetric. The considered norm of B is the maximum between the norm as an operator from $\mathbf{l}^+$ to itself and as an operator from $\mathbf{l}^-$ to itself.

So one has

$$\left\{f^{(1)}; g^{(2)}\right\}^{(1)} = \left\langle f^1(I,\alpha), JB(I,\alpha)\xi\right\rangle = -\left\langle B(I,\alpha)Jf^1(I,\alpha),\xi\right\rangle \ . \tag{A.7}$$

From this formula one has that the ξ component of the vector field, given by $-JB(I,\alpha)f^1(I,\alpha)$ is actually estimated by (3.26).

We come to the α component of the vector field: it is given by

$$-\left\langle \frac{\partial B}{\partial I}Jf^1,\xi\right\rangle - \left\langle \frac{\partial f^1}{\partial I}, JB\xi\right\rangle \ .$$

We start by estimating the second term. To this end remark that

$$\frac{\partial f^1}{\partial I}(I,\alpha) = \nabla_\xi\left[X_{f^{(1)}}\right]_\alpha(I,\alpha,0) \ ,$$

so that, exploiting the fact that $\mathbf{l}^+$ is the dual of $\mathbf{l}^-$, its norm (3.13) can be bounded using Cauchy inequality:

$$\left\langle\left|\frac{\partial f^1}{\partial I}(I,\alpha)\right|\right\rangle_+ \le \frac{1}{R(1-d)}\sup_{\mathcal{G}^-_d}\left|\left[X_{f^{(1)}}\right]_\alpha\right| \le \frac{1}{1-d}N^{\mathcal{S}}_d\left(f^{(1)}\right)R_\alpha \ .$$

Using also the estimate (A.6) one thus gets

$$\frac{1}{R_\alpha}\left|\left\langle \frac{\partial f^1}{\partial I}, JB\xi \right\rangle\right| \le \frac{1}{(1-d)} N_d^{\mathcal{S}}\left(f^{(1)}\right)\|B\| R(1-d)$$
$$\le R N_d^{\mathcal{S}}\left(f^{(1)}\right)\frac{1}{1-d}N_d^{\nabla}\left(g^{(2)}\right) \ ,$$

dividing by R one gets the wanted estimate. We now estimate the term involving the derivative of B. Remark first that one has

$$[X_{g_2}]_\alpha = \left\langle \xi ; \frac{\partial B}{\partial I}\xi \right\rangle \ ,$$

so that the norm of $\frac{\partial B}{\partial I}$ as an operator from $\mathbf{l}^-$ to $\mathbf{l}^+$ is estimated by

$$\left\|\frac{\partial B}{\partial I}\right\| \le \frac{2}{R^2(1-d)^2}\sup_{\mathcal{G}_d^-}\left\|[X_{g_2}]_\alpha\right\| \le \frac{2}{R(1-d)^2}N_d^{\nabla}(g_2) \ ,$$

from which, on $\mathcal{G}_d^-$,

$$\frac{1}{R_\alpha}\left|\left\langle \frac{\partial B}{\partial I}Jf^1, \xi \right\rangle\right| \le \frac{2}{R(1-d)^2}N_d^{\nabla}(g_2)\, R(1-d)\sup_{\mathcal{G}_d^-}\left\|\left[X_{f^{(1)}}\right]_\xi\right\|_+$$
$$\le \frac{2R}{(1-d)}N_d^{\nabla}(g_2)\, N_d^{\mathcal{S}}\left(f^{(1)}\right) \ .$$

Collecting the results the thesis follows. □

Lemma A.4. *Let $\chi \in \mathcal{S}_d$, with $0 \le d_1 < 1-d$ and let $f \in \mathcal{A}_d$; fix $0 < d_1 < (1-d)$, assume $N_d^{\mathcal{S}}(\chi) \le d_1/3$, then, for $|t| \le 1$, one has*

$$N_{d+d_1}^{\nabla}\left(f\circ\Phi_\chi^t\right) \le \left(1 + \frac{3}{d_1}N_d^{\mathcal{S}}(\chi)\right)N_d^{\nabla}(f) \ .$$

If $f \in \mathcal{S}_d$ then the same estimate holds in the norm $N_{\cdot}^{\mathcal{S}}(.)$.

Proof. In this proof we omit the index χ from Φ. First remark that, since Φ^t is a canonical transformation one has

$$X_{f\circ\Phi^t}(\zeta) = \mathrm{d}\Phi^{-t}(\Phi^t(\zeta))X_{f(\Phi^t(\zeta))} \ , \tag{A.8}$$

from which

$$X_{f\circ\Phi^t}(\zeta) = \left(\mathrm{d}\Phi^{-t}(\Phi^t(\zeta)) - \mathbb{1}\right)X_{f(\Phi^t(z))} + X_{f(\Phi^t(z))} \ .$$

We first estimate $X_{f\circ\Phi^t}$ in $\mathbf{l}^+$. To estimate the first term fix $\bar{d} := d_1/3$; we have

$$\sup_{\zeta\in\mathcal{G}_{3\bar{d}}^+}\left\|\mathrm{d}\Phi^{-t}(\Phi^t(\zeta)) - \mathbb{1}\right\| \le \sup_{\zeta\in\mathcal{G}_{2\bar{d}}^+}\left\|\mathrm{d}\Phi^{-t}(\zeta) - \mathbb{1}\right\| \tag{A.9}$$
$$\le \frac{1}{\bar{d}}\sup_{\zeta\in\mathcal{G}_{\bar{d}}^+}\left\|\Phi^{-t}(\zeta)-\zeta\right\|_+ \le \frac{1}{\bar{d}}N_d^{\mathcal{S}}(\chi) \ ,$$

where the differential of $\Phi^{-t}(\zeta)$ is considered as an operator from $\mathbf{l}^{+}$ to $\mathbf{l}^{+}$. Going back to d_1, adding the trivial estimate of the second term and the estimate in $\mathbf{l}_{-}$, one gets the thesis. $\square$

Proof of Lemma 3.11. First define $\bar{d} := \frac{d}{2(N+1)}$. Using (3.23) l-times, one has

$$N^{\mathcal{S}}_{d+l\bar{d}}\left(f_{(l)}\right) \leq \left(\frac{2}{\bar{d}} N^{\mathcal{S}}_{d}\left(\chi\right)\right)^{l} N^{\nabla}_{d}\left(f\right) \ .$$

By Lemma A.4 one has

$$N^{\mathcal{S}}_{d+d_1}\left(f_{(N+1)} \circ \Phi^{t}\right) \leq \left(1 + \frac{6}{d_1} N^{\mathcal{S}}_{d}\left(\chi\right)\right) \left(\frac{4(N+1)}{d_1} N^{\mathcal{S}}_{d}\left(\chi\right)\right)^{N+1} N^{\nabla}_{d}\left(f\right) \ ,$$

which gives the thesis. $\square$

Proof of lemma 3.12. We start by $\chi^{(0)}$. It is well known (see e.g. [BG93]) that defining

$$\chi^{(0)}(I,\alpha) := \frac{1}{2\pi\omega(I)} \int_{0}^{2\pi} t\Psi^{(0)}(I,\alpha+t)\mathrm{d}t \ , \tag{A.10}$$

it solves the equation

$$\left\{H_{lin};\chi^{(0)}\right\} = \Psi^{(0)} \ .$$

Then one has

$$X_{\chi^{(0)}}(I,\alpha) := \frac{1}{2\pi\omega(I)} \int_{0}^{2\pi} tX_{\Psi^{(0)}}(I,\alpha+t)\mathrm{d}t \ , \tag{A.11}$$

from which the estimate (3.33) immediately follows.

We now study the equation

$$\left\{H_{lin};\chi^{(1)}\right\} = \Psi^{(1)} \ ; \tag{A.12}$$

inserting the decomposition (3.20), i.e. writing

$$\Psi^{(1)}(I,\alpha,z,w) = \langle \Psi^{1}_{z}; z\rangle + \langle \Psi^{1}_{w}; w\rangle \ , \tag{A.13}$$

and similarly for χ, one gets that (A.12) is equivalent to the couple of equations

$$\omega(I)\frac{\partial \chi^{1}_{z}}{\partial \alpha} - \mathrm{i}\chi^{1}_{z} = \Psi^{1}_{z} \ , \quad \omega(I)\frac{\partial \chi^{1}_{w}}{\partial \alpha} + \mathrm{i}\chi^{1}_{w} = \Psi^{1}_{w} \ . \tag{A.14}$$

Let's focus on the second one. Component wise this is an ordinary differential equation in the independent variable α, which can be easily solved by Duhamel formula. Imposing the solution to be periodic of period 2π in α one gets a unique solution given by

$$\chi^{1}_{w}(I,\alpha) = \frac{1}{\omega\left(\mathrm{e}^{\mathrm{i}2\pi\frac{1}{\omega}} - 1\right)} \int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}2\pi\frac{1}{\omega}s}\Psi^{1}_{w}(I,\alpha+s)\mathrm{d}s \ . \tag{A.15}$$

For χ^{1}_{z} one gets an identical formula with -1 in place of 1. From (A.15) one gets identical formulae for the functions $\langle\chi^{1}_{w}, w\rangle$, $\langle\chi^{1}_{z}, z\rangle$ and for their Hamiltonian vector fields. Inserting the corresponding estimates of the vector field of $\langle\Psi^{1}_{w}, w\rangle$ and of the other functions and computing the integrals one gets the thesis. $\square$

B Technical lemmas for the dispersive estimates

Lemma B.1. *Let $\varphi(\theta)$ be a function of class $C^k(a,b)$, $k \geq 2$ and assume that*

$$\left|\varphi^{(k)}(\theta)\right| \geq \delta_k > 0 \ , \quad \forall \theta \in (a,b) \ ,$$

then there exists c_k s.t.

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}\lambda\varphi(\theta)} \mathrm{d}\theta\right| \leq \frac{c_k}{|\lambda\delta_k|^{1/k}} \ . \tag{B.1}$$

The proof is a minor variant of the proof of Proposition 2 p.332 of [Ste93] (Van der Corput Lemma), and is omitted.

We now recall the properties of $-\Delta$ and in particular the Puiseaux expansion for $R_{-\Delta}$ proved in [KKK06] and specialize it to skew symmetric sequences.

By using an explicit computation and the Cauchy formula for the computation of integrals [KKK06], proved the following lemma:

Lemma B.2. *(2.1 of [KKK06])For $\tilde{\nu} \in \mathbb{C} - [0,4]$ the Kernel of the resolvent of $-\Delta$ is given by*

$$R_{-\Delta}(\tilde{\nu}, j, k) = -\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\theta(\tilde{\nu})|j-k|}}{2\sin(\theta(\tilde{\nu}))} \tag{B.2}$$

where $\theta(\tilde{\nu})$ is the unique solution of the equation

$$2 - 2\cos\theta = \tilde{\nu}$$

in the domain $\{-\pi \leq \mathrm{Re}\,\theta \leq \pi \ ; \ \mathrm{Im}\,\theta < 0\}$.

By this we mean that

$$(R_{-\Delta}(\tilde{\nu})q)_j = \sum_k R_{-\Delta}(\tilde{\nu}, j, k) q_k \ .$$

Corollary B.3. *Equation* (4.12) *holds for $\nu \in \mathbb{C} - [1, 1+4\epsilon]$.*

In Lemma 3.1 of [KKK06], by direct computation of the limit of (B.2), it is shown that the limit

$$\lim_{\epsilon \to 0^+} R_{-\Delta}(\tilde{\nu} \pm \mathrm{i}\epsilon) = R^{\pm}_{-\Delta}(\tilde{\nu}) \ , \quad \tilde{\nu} \in (0,4) \tag{B.3}$$

exists in $B(\ell^2_s, \ell^2_{-s})$ for all $s > 1/2$, and this implies a similar result for R_B.

Remark that the term proportional to $|\tilde{\nu}|^{-1/2}$ in our case is missing. This is due to the fact that its coefficient is proportional to $\sum_l q_l$, which vanishes for skewsymmetric sequences.

Then we need the following lemma, which is a particular case of Lemma 3.2 of [KKK06].

Lemma B.4. *Let $s > 3/2$, then for $\tilde{\nu} \to 0$ one has the following asymptotic expansion, valid for* **skew-symmetric sequences** q

$$\left[R^{\pm}_{-\Delta_1}(\tilde{\nu})q\right]_k = -\frac{1}{2}\sum_l |k-l|\, q_l + r(\tilde{\nu})q \ , \tag{B.4}$$

where $\|r(\tilde{\nu})\|_{B(\ell^2_s,\ell^2_{-s})} = O(|\tilde{\nu}|^{1/2})$ and, for $s > \frac{1}{2}+i$, $i \geq 1$ one has

$$\left\|\frac{d^i}{d\tilde{\nu}^i} R^{\pm}_{-\Delta}\right\|_{B(\ell^2_s,\ell^2_{-s})} = O(|\tilde{\nu}|^{\frac{1}{2}-i}) \ . \tag{B.5}$$

A similar expansion holds for $\tilde{\nu} \to 4$.

We also need the following Lemma by Jensen-Kato

Lemma B.5. *Let $\mathcal{B}$ be a Banach space, and let $F \in C^2((0,a),\mathcal{B})$, assume*

$$F(0) = F(a) = 0 \ , \quad \left\|\frac{d^i}{d\tilde{\nu}^i}F(\tilde{\nu})\right\|_{\mathcal{B}} \leq C|\tilde{\nu}|^{\frac{1}{2}-i} \ , \quad \tilde{\nu} \to 0 \ , \quad i = 1,2$$

then for any $|t| > 1$ one has

$$\left|\int_0^a \mathrm{e}^{\mathrm{i}t\tilde{\nu}} F(\tilde{\nu})\mathrm{d}\tilde{\nu}\right| \leq \frac{C}{|t|^{3/2}} \tag{B.6}$$

For the proof see [JK79] (see also [PS08]).

Lemma B.6. *One has*

$$\|q\|_{L^2_t\ell^\infty_{-s}} \preceq \|q\|_{\ell^\infty_{-s'},L^2_t} \ , \quad \forall s > s' + \frac{1}{2} \ .$$

Proof. One has

$$\begin{aligned}
\|q\|^2_{L^2_t\ell^\infty_{-s}} &= \int \left[\sup_n \langle n\rangle^{-s}|q_n(t)|\right]^2 \mathrm{d}t \\
\leq \int \sum_n \langle n\rangle^{-2s}|q_n(t)|^2 \mathrm{d}t &= \sum_n \langle n\rangle^{-2s} \int |q_n(t)|^2\, \mathrm{d}t \\
\leq \left[\sum_n \langle n\rangle^{-2(s-s')}\right] &\sup_n \left[\langle n\rangle^{-2s'} \int |q_n(t)|^2\, \mathrm{d}t\right] \\
&= C\, \|q\|^2_{\ell^\infty_{s'} L^2_t}
\end{aligned}$$

□

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